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PVA EXPO PRAGUE

Solutions

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Solutions of problems

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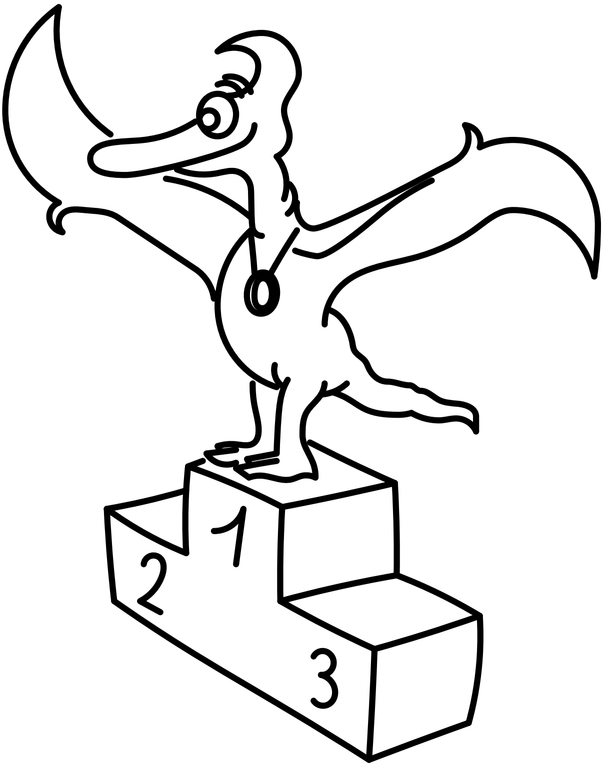

Problem AA ... summer drink

Consider a cylindrical glass with an internal radius of $R = 3.0\,\mathrm{cm}$ *and an ice cube with a side length of* $a = 2.5\,\mathrm{cm}$ *floating on the surface. The height of the liquid level, including the floating ice, is* $h = 10\,\mathrm{cm}$*. By how many centimeters will the level change once the ice cube has completely melted? Provide a positive value if the level increases and a negative value if it decreases.* *Vlado thought that he got lemonade with more ice than water. He was wrong.*

The dimensions of the cube are much smaller than the height of the liquid level; therefore, the cube will float freely on the surface. A floating cube is thus in equilibrium, meaning that the resultant force acting on it is zero. According to Archimedes' principle, we have

$$mg = \rho_\mathrm{v} V_\mathrm{p} g \quad \Rightarrow \quad m = \rho_\mathrm{v} V_\mathrm{p}\,,$$

where m is the mass of the ice cube and V_p is the volume of its submerged part. After placing the cube into the glass of water, the water level rises by

$$\Delta h_0 = \frac{V_\mathrm{p}}{\pi R^2}\,,$$

from which it follows that the height of the water level before inserting the ice cube was $h_0 = h - \Delta h_0$.

During the melting of the cube, its mass is conserved; therefore, the water level increases by

$$\Delta h_1 = \frac{\frac{m}{\rho_\mathrm{v}}}{\pi R^2} = \frac{\frac{\rho_\mathrm{v} V_\mathrm{p}}{\rho_\mathrm{v}}}{\pi R^2} = \frac{V_\mathrm{p}}{\pi R^2} = \Delta h_0\,.$$

The height of the water level after the cube has melted is

$$h_1 = h_0 + \Delta h_1 = h - \Delta h_0 + \Delta h_0 = h\,.$$

The height of the water level, therefore, clearly does not change ($\Delta h = 0$). This result holds in general for an ice "cube" of arbitrary shape, since we did not use any specific geometric properties associated with a cube in the calculation.

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Problem AB ... Marek's ride

A sightseeing tram T3 Coupé *equipped with turbo engines is racing through the streets at breakneck speed. Behind the controls sits nobody other than he. The man. The myth. The legend. Marek Milička. With every passing second, he approaches a collapsed bridge over the Vltava river, which has a width of* $d = 200\,\mathrm{m}$*. The remains of the bridge consist of ramps on both banks, which form an angle of* $\vartheta = 10^\circ$ *with the ground. Marek accelerates and... he jumps over the river with the tram, landing successfully on the ramp on the opposite bank. At what speed was Marek's tram moving immediately before the jump?*

Even such things might happen at a FYKOS camp.

Denote the initial speed of the tram as v. Since it moves along a ramp, the velocity vector forms an angle ϑ with the horizontal plane. The horizontal (x) and vertical (y) components of the velocity are then

$$v_x = v\cos\vartheta\,,$$
$$v_y = v\sin\vartheta\,.$$

Let t be the time it takes for the tram to jump over the river. Then we have

$$vt\cos\vartheta = d\,.$$

Moreover, in the vertical direction, the tram will be decelerated by gravity g during the first half of the time until it reaches zero vertical speed, after which it will be accelerated back downward to land on the other side with the same vertical speed but in the opposite direction. Thus,

$$v\sin\vartheta - g\frac{t}{2} = 0\,.$$

From this equation, we can solve for t and substitute it into the previous equation:

$$t = 2\frac{v}{g}\sin\vartheta \quad\Rightarrow\quad 2\sin\vartheta\cos\vartheta\frac{v^2}{g} = d\,.$$

Using the identity $\sin 2\vartheta = 2\sin\vartheta\cos\vartheta$ and rearranging, we obtain for the speed v

$$v = \sqrt{\frac{gd}{\sin 2\vartheta}} \doteq 75.7\,\mathrm{m{\cdot}s^{-1}} \doteq 272.7\,\mathrm{km{\cdot}h^{-1}}\,.$$

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Problem AC ... don't judge a fish by its ability to run

A rabbit and a fish compete in a race in which the rabbit runs and the fish swims. They start at the same time and travel a distance of $s = 500\,\mathrm{m}$ downstream. Then they turn around and return to the starting point. The river has a current speed of $u = 1.0\,\mathrm{m{\cdot}s^{-1}}$. The rabbit runs at the same speed as the fish swims, which is $v = 10\,\mathrm{m{\cdot}s^{-1}}$. What is the difference in their finishing times? Provide a positive value if the rabbit finishes first, and a negative value if the fish finishes first. *Lego taught about motion in a medium.*

The hare's time will be simply $2s/v = 100\,\mathrm{s}$. The fish will have a speed downstream of $v + u$, giving a time for this half of the distance of $s/(v+u) \doteq 45.45\,\mathrm{s}$. Upstream, its speed is $v-u$, so the time is $s/(v-u) \doteq 55.55\,\mathrm{s}$. Altogether, it will take the fish approximately $101.0\,\mathrm{s}$, meaning it arrives $1.0\,\mathrm{s}$ later than the hare. Since we are to provide a positive result if the hare arrives first, the answer is therefore $1.0\,\mathrm{s}$.

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Problem AD ... angle averaging

Lego had an unpleasant dream: he found himself in the middle of a herd of running horses. He reasoned that the risk of injury would be minimal if he ran approximately in the same direction as they did. He therefore wanted to determine their average direction. Using a reference vector, he measured the directions of several nearby horses relative to it, obtaining the angles $1^\circ, 5^\circ, 2^\circ, 358^\circ, 357^\circ$. At what angle with respect to this vector should Lego run so that it is the average direction of the horses in his vicinity?

While doing research, Lego read about flocking.

If we were to take the arithmetic mean, we would obtain: $(1^\circ + 5^\circ + 2^\circ + 358^\circ + 357^\circ)/5 = 144.6^\circ$. This result does not make sense, since all horses are running approximately in the direction of our vector, yet the average direction comes out as 144.6°. This is, of course, caused by the discontinuity at zero (where the angles 0° and 360° are equivalent).

We can resolve it by removing this jump and taking the angles from the interval $(-180^\circ, 180^\circ)$. Then we obtain the average $(1^\circ + 5^\circ + 2^\circ + (-2^\circ) + (-3^\circ))/5 = 0.6^\circ$, which is already a plausible value (and sufficient for the purposes of our problem).

The procedure from the previous paragraph works only in the case when the directions are indeed close to one another. If, for example, we had a significantly larger data set covering various directions more uniformly, we could no longer average them so simply. In such a case, the following procedure is used: we compute the average sine and the average cosine of the angles (these functions are continuous and have no jumps), and we obtain the resulting mean angle as the arctangent of their ratio. In our case, it would read like this:

$$\arctan \frac{\frac{1}{5}(\sin 1^\circ + \sin 5^\circ + \sin 2^\circ + \sin 358^\circ + \sin 357^\circ)}{\frac{1}{5}(\cos 1^\circ + \cos 5^\circ + \cos 2^\circ + \cos 358^\circ + \cos 357^\circ)} \doteq 0.6^\circ \,.$$

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Problem AE ... vive la révolution

The guillotine has a blade of mass m that slides along the grooves of two vertical, opposing beams as it falls. The coefficient of friction between the blade and the beams is k, and the force between each beam and the blade is F. An energy E is required to sever the head. Determine the minimum height of the guillotine when accounted for energy lost due to friction.

Peter studied the history of France and watched a guillotine demonstration.

According to the problem statement, all energy expended in slowing down the blade due to friction is converted into heat. Therefore, the total required initial potential energy will equal the sum of the desired kinetic energy E and the work done by friction. This work can be calculated as the sum of the frictional forces kF on both beams multiplied by the distance h over which the forces act. Altogether, this gives $2kFh$. Note that, although the normal force F acts on each of the two grooves on opposite sides of the blade, the resulting prefactor is only 2, not 4, because the frictional force does not depend on the contact area. Now it is sufficient to write the required potential energy of the blade mgh in the equation

$$\begin{aligned} mgh &= E + 2kFh\,, \\ (mg - 2kF)\,h &= E\,, \\ h &= \frac{E}{mg - 2kF}\,. \end{aligned}$$

Therefore, the guillotine must be at least $h = E/(mg - 2kF)$ high.

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Problem AF ... nacho dip

We have a pile of nachos and a big bowl of dip. The nachos have the shape of an equilateral triangle with side length $a = 5.2\,\mathrm{cm}$. By what percentage will we consume more dip if we dip the nachos while holding them by a vertex compared to holding them by the midpoint of a side? In both cases, we hold the nachos so that our fingers remain clean, leaving a length $h = 0.62\,\mathrm{cm}$ undipped. Assume that a constant thickness of dip is collected. Neglect the thickness of the nachos. *Karel was thinking about everything he can't eat.*

Compare the two cases—in the first one, we hold the triangle (nacho) vertically by its vertex; in the second, we hold it by the midpoint of one side. In both cases, we want to calculate the area of the part dipped into the dip.

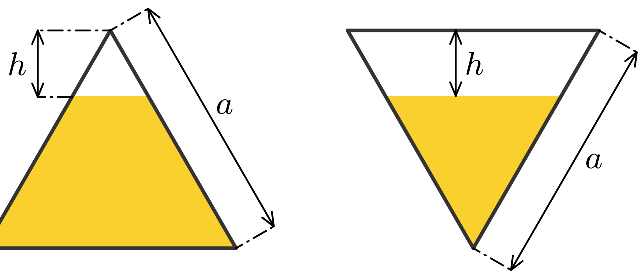

Figure 1: Geometry of the nachos after dipping into the sauce in the first (left) and second (right) case.

Of course, the triangle is dipped on both sides, but since we are interested only in the ratio of the areas, we will consider only one side of the triangle. Recall that the height of an equilateral triangle with side a is $a\sin(60^\circ) = a\sqrt{3}/2$, so its area is $a\sqrt{3}/2 \cdot a/2 = a^2\sqrt{3}/4$.

In the first case, the entire triangle (with side a) is dipped except for a smaller equilateral triangle of height h, whose side is $2h/\sqrt{3}$. Therefore, the area of the dipped part is

$$S_1 = \frac{\sqrt{3}}{4}a^2 - \frac{\sqrt{3}}{4}\left(\frac{2h}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{4}a^2 - \frac{\sqrt{3}}{3}h^2\,.$$

In the second case, only a portion of the triangle is dipped, shaped as an isosceles triangle with height $a\sqrt{3}/2 - h$. Its side is therefore $2(a\sqrt{3}/2 - h)/\sqrt{3}$, giving the corresponding area

$$S_2 = \frac{\sqrt{3}}{4}\left(\frac{2\left(\frac{\sqrt{3}}{2}a - h\right)}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{4}a^2 - ah + \frac{\sqrt{3}}{3}h^2\,.$$

Since we are interested in the percentage by which more dip is used in the first case, we subtract 100 % from the ratio S_1/S_2. The resulting expression is

$$\frac{S_1 - S_2}{S_2} = \frac{ah - 2\frac{\sqrt{3}}{3}h^2}{\frac{\sqrt{3}}{4}a^2 - ah + \frac{\sqrt{3}}{3}h^2} = \frac{4h}{\sqrt{3}a - 2h} \doteq 32\,\%\,.$$

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Problem AG ... active power plant

The Temelín nuclear power plant has a thermal power output of $P = 6.2\,\mathrm{GW}$. Assume that the average usable energy released in the decay of one nucleus of uranium $^{235}\mathrm{U}$ is $E_0 = 200\,\mathrm{MeV}$, and that the entire power output comes from this decay. The molar mass of the isotope $^{235}\mathrm{U}$ is $M_{\mathrm{U235}} = 235\,\mathrm{g{\cdot}mol^{-1}}$, and its half-life is $T = 7.04\cdot 10^8$ years. The activity of one banana is $A = 15\,\mathrm{Bq}$. How many times greater is the activity of the $^{235}\mathrm{U}$ consumed in $t = 1.0\,\mathrm{s}$ at the Temelín nuclear power plant compared to a banana?

This problem is brought to you by the CEZ Group.

David wanted to be active, but he didn't feel like exercising.

First, we calculate how many uranium $^{235}\mathrm{U}$ nuclei decay in the Temelín nuclear power plant. In one second, the number of decays will be

$$N = \frac{Pt}{E_0} \approx 2.1\cdot 10^{20}\,.$$

We pay attention to the units and note that $E_0 = 200\,\mathrm{MeV} = 3.20\cdot 10^{-11}\,\mathrm{J}$. For the activity, the formula

$$A = \lambda N$$

holds, where λ is the decay constant for which the following holds

$$\lambda = \frac{\ln 2}{T}\,,$$

where T is the half-life.

By combining these formulas and using the fact that we are interested in the ratio of the activities of the Temelín nuclear power plant A_{U} and a banana A, we obtain the final result as

$$\frac{A_{\mathrm{U}}}{A} = \frac{\frac{\ln 2}{T}\frac{Pt}{E_0}}{A} = \frac{Pt\ln 2}{AE_0T} = \frac{6.2\cdot 10^9\,\mathrm{W}\cdot 1\,\mathrm{s}\cdot\ln 2}{15\,\mathrm{Bq}\cdot 3.20\cdot 10^{-11}\,\mathrm{J}\cdot 2.22\cdot 10^{16}\,\mathrm{s}} \doteq 402\,.$$

The activity of the uranium consumed in the Temelín nuclear power plant is about 402 times greater than the activity of the banana.

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Problem AH ... water in the air

A firefighter is extinguishing a fire by continuously spraying the surroundings with a steady stream of water from a hose from an elevated position. The hose is held so that the nozzle is at a height $h_0 = 3.2\,\mathrm{m}$ above the ground, onto which the water subsequently falls. Water flows out of the hose of diameter $d = 75\,\mathrm{mm}$ with speed $v_0 = 4.2\,\mathrm{m\cdot s^{-1}}$ at an initial angle $\alpha = 35^\circ$ upward relative to the horizontal plane. What is the mass of water that is present in the air at any given moment? *Karel was thinking about the firefighters and the matriculation water arch.*

To determine the volume of water present in the air at any given moment, we first need to calculate how long each water element remains in the air. Its initial height above the ground is h_0, and its initial vertical velocity is $v_{0y} = v_0 \sin\alpha$, so its instantaneous height above the ground is $h(t) = h_0 + v_{0y}t - gt^2/2$. At the moment when the water element reaches the ground, this height must be equal to zero; it therefore suffices to solve the resulting equation with zero on the right-hand side for the time t

$$h_0 + v_{0y}t - \frac{1}{2}gt^2 = 0 \quad\Rightarrow\quad t = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2gh_0}}{g} \doteq 1.09\,\mathrm{s}\,,$$

where we have chosen the plus sign because we are interested in a positive value of the time t (a time in the future). Now that we know the time t, it suffices to determine how much water enters the air during this time, that is, how much is expelled from the hose. The hose has a flow speed v_0 and a cross-sectional area $S = \pi d^2/4$, so during the time t, a volume $V = Sv_0t$ flows through it. This volume must be multiplied by the density of water ρ, yielding the desired mass of water present in the air

$$M = \rho V = \rho S v_0 t = \frac{\pi d^2 \rho v_0}{4} \frac{v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2gh_0}}{g} \doteq 20\,\mathrm{kg}\,.$$

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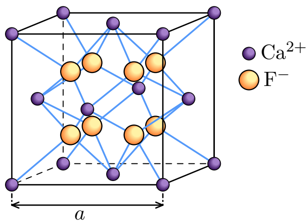
Problem BA ... Valentine's fluorite

As Vlado searched for a romantic gift for his girlfriend Julka before Valentine's Day, he visited a store with various decorative minerals. He was attracted to fluorite products and thus began to ponder their mystical properties. Calculate the density of a fluorite single crystal and express the result to four significant figures. One unit cell of fluorite CaF_2 has a dimension $a = 5.463\,\mathrm{Å}$. Its crystal lattice is shown in the attached figure. The molar mass of calcium and fluor is $M_{\mathrm{Ca}} = 0.040\,08\,\mathrm{kg\cdot mol^{-1}}$ and $M_{\mathrm{F}} = 0.019\,00\,\mathrm{kg\cdot mol^{-1}}$, respectively.



This year, it will be flowers again.

A monocrystalline fluorite crystal is formed by the repetition of a large number of unit cells. By counting the "spheres in the figure", we find that the unit cell consists of 8 fluorine atoms and 14 calcium atoms; however, we must realize that each calcium atom located at the center of a face is shared by two cells and that each calcium atom located at a vertex is shared among eight cells. Altogether, we therefore find that, on average, one cell in the entire crystal contains 8 fluorine atoms and $1 + 4/2 + 8/8 = 4$ calcium atoms—this also corresponds to the fact that the empirical formula of fluorite is CaF_2.

To compute the density, it now suffices to determine the ratio of the mass of these 12 atoms to the volume of the unit cell, that is,

$$\rho = \frac{m}{V} = \frac{8m_\mathrm{F} + 4m_\mathrm{Ca}}{a^3} = \frac{8M_\mathrm{F} + 4M_\mathrm{Ca}}{N_\mathrm{A} a^3}\,,$$

where $N_\mathrm{A} = 6.022 \cdot 10^{-23}\,\mathrm{mol^{-1}}$ denotes Avogadro's constant and $M_\mathrm{Ca} = 0.040\,08\,\mathrm{kg{\cdot}mol^{-1}}$ and $M_\mathrm{F} = 0.019\,00\,\mathrm{kg{\cdot}mol^{-1}}$ denote the molar masses of the individual elements. Converting $a = 5.463\,\mathrm{Å} = 5.463 \cdot 10^{-10}\,\mathrm{m}$ and substituting, we obtain the result

$$\rho \doteq 3\,181\,\mathrm{kg{\cdot}m^{-3}}\,,$$

which agrees with the tabulated value.

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Problem BB ... overturned glass

Imagine a cylindrical glass of height h and base diameter d. Suppose that its center of mass is at the center of its axis. What is the minimum coefficient of static friction with the surface required to make it possible to tip the glass over solely by pushing from the side? The push may be applied at any point, but only horizontally.

Lego broke a glass.

The torque that we must overcome is the torque by which gravity would act (in the case that the cup is at the tipping threshold) with respect to the point of contact with the surface. Gravity acts at the center of the cup, so relative to the point of contact, it has a horizontal distance (lever arm) $d/2$; this torque is therefore $M_1 = mgd/2$, where we have denoted the mass of the cup by m.

If we were to push the cup with a force greater than the maximum static friction force, it would start sliding along the surface, and the coefficient would simultaneously drop to the coefficient of kinetic friction. The greatest chance to tip the cup over is therefore when we push with exactly the static friction force, that is, fmg.

At the same time, it matters where we push it. If we were to push the cup at its base, it is probably intuitive that we would not tip it over. We produce the largest torque when we push at its highest point, that is, at a height h above the base. Then the pushing force and the friction force together act on the cup with a torque $M_2 = mgfh$. We obtain the equation

$$M_1 = M_2\,,$$

$$mg\frac{d}{2} = mgfh\,,$$

$$\frac{d}{2h} = f\,.$$

This is therefore the minimum f required to start tilting the cup. However, once we tilt it, the vertical distance of the point where we push relative to the axis of rotation increases, while at the same time, the horizontal distance of the center of mass from the axis of rotation decreases. As a result, the f that was sufficient to tilt the cup at least slightly will certainly be sufficient to overturn the cup completely.

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Problem BC ... coin on water

A metal coin with density $\rho = 2\,580\,\mathrm{kg{\cdot}m^{-3}}$, a radius $r = 8.10\,\mathrm{mm}$ and a height $h = 0.651\,\mathrm{mm}$ is lying on water. Due to surface tension, the coin remains afloat. To what minimum temperature must the water be heated for the coin to sink? The surface tension of water at a temperature of $50.0\,\mathrm{^\circ C}$ is $\sigma_{50} = 67.92\,\mathrm{mN{\cdot}m^{-1}}$ and at a temperature of $60.0\,\mathrm{^\circ C}$ it is $\sigma_{60} = 66.18\,\mathrm{mN{\cdot}m^{-1}}$. Assume that the dependence of surface tension on temperature is linear.

Danka would like to try lying down on the water and not getting wet.

The resultant force caused by the surface tension of water acting on the coin is $F = -\sigma l \cos\theta$, where $l = 2\pi r$ is the length over which the coin is in contact with the water surface and θ is the contact (wetting) angle of the water surface relative to the edge of the coin. For the coin to float on the surface, the force due to surface tension must balance the weight of the coin, so the following must hold

$$-2\pi\sigma r\cos\theta = mg\,,$$

where $m = \pi r^2 h\rho$. Now it is clear that the limiting case with the minimum sufficient surface tension corresponds to the situation $\theta = 180^\circ$. From this, we obtain the value of the surface tension

$$\sigma = \frac{mg}{2\pi r} = \frac{rh\rho g}{2}\,.$$

The only thing left to do is to calculate the temperature at which water attains this value of surface tension. If we denote $t_{50} = 50.0\,\mathrm{^\circ C}$ and $t_{60} = 60.0\,\mathrm{^\circ C}$, the linear dependence described by the values σ_{50} and σ_{60} can be written as $\sigma(t) = \sigma_{50} + (\sigma_{60} - \sigma_{50})(t - t_{50})/(t_{60} - t_{50})$. From this expression, we solve for the temperature t, which yields step by step

$$\sigma_{50} + (\sigma_{60} - \sigma_{50})\frac{t - t_{50}}{t_{60} - t_{50}} = \frac{rh\rho g}{2}\,,$$

$$\frac{t - t_{50}}{t_{60} - t_{50}} = \frac{\frac{rh\rho g}{2} - \sigma_{50}}{\sigma_{60} - \sigma_{50}}\,,$$

$$t = t_{50} + \frac{\frac{rh\rho g}{2} - \sigma_{50}}{\sigma_{60} - \sigma_{50}}(t_{60} - t_{50}) \doteq 56.8\,\mathrm{^\circ C}\,.$$

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Problem BD ... prone to procrastination

Duty calls, so Marek grabs it, ties it to a rope of length $l = 3.0\,\mathrm{m}$, spins it so that the rope makes an angle $\alpha = 15^\circ$ with the horizontal plane, and releases it from the rope. However, since duty is persistent, it starts accelerating from rest toward Marek at $a = 1.5\,\mathrm{m{\cdot}s^{-2}}$ the moment it lands, while Marek has been running away from the impact point with a speed of $v = 15\,\mathrm{km{\cdot}h^{-1}}$

ever since he released the rope. If Marek is $h = 2.0\,\mathrm{m}$ tall and holds the rope at this height, how long after landing will duty catch up with him?

Marek is a man of focus, commitment, and sheer will.

The problem involves a series of intricate calculations that build on one another.

Let us first analyze the situation when Marek was spinning the duty. From the force balance, we find that the centrifugal force acting on duty was $F_o = mg\cot\alpha = mv_\mathrm{t}^2/r$, where m is the mass of duty, g is the gravitational acceleration, v_t is the tangential speed of duty, and r is the radius of rotation, for which $r = l\cos\alpha$ holds. We thus obtain the initial speed of the subsequent horizontal throw as

$$v_\mathrm{t} = \sqrt{gl\cos\alpha\cot\alpha} \doteq 10.3\,\mathrm{m{\cdot}s^{-1}}\,.$$

Its initial height was $h' = h - l\sin\alpha$, and the motion lasted for a time $t = \sqrt{2h'/g} \doteq 0.50\,\mathrm{s}$. During this time, duty traveled a distance $D = v_\mathrm{t}t \doteq 5.14\,\mathrm{m}$, but because it did not move directly away from Marek, a top view reveals a right triangle with legs r and D. The sought distance of duty's landing point from Marek is therefore

$$d = \sqrt{D^2 + r^2} \doteq 5.90\,\mathrm{m}\,.$$

At the moment when Marek released duty, he started running; therefore, when duty starts running after him, Marek has an initial lead

$$x_0 = vt + d \doteq 7.98\,\mathrm{m}\,.$$

For the time T after which duty catches up with Marek, the following holds:

$$\frac{1}{2}aT^2 = x_0 + vT\,,$$
$$T = \frac{v + \sqrt{v^2 + 2a\,x_0}}{a}\,,$$

where we have taken the positive root that makes physical sense.

By substitution into the formulas, we obtain $T = 7.1\,\mathrm{s}$.

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Problem BE ... Schrödinger's cat rescue mission

Anička keeps Schrödinger's cat in a closed box and initially assigns a probability $p_0 = 50\%$ that the cat is alive. She worries about the cat, so she makes a deal with a fairy; the fairy casts a resurrection spell three times vertically downward into the box. Each spell hits the box at a uniformly random position, independently of the others. The box is rectangular with side lengths $a = 90\,\mathrm{cm}$ and $b = 75\,\mathrm{cm}$. In top view, the cat can be modeled as a circle of radius $r = 15\,\mathrm{cm}$, placed at a uniformly random position within the box, that remains stationary during the casting. If any cast spell hits the cat, it is considered saved. What is the probability of the cat being alive after the three shots?

Anička is afraid of cats.

It is more convenient to calculate the probability p_n that the cat is not hit—this allows us to avoid dealing with the case where the cat is hit multiple times. The probability that the cat is hit is then $p_\mathrm{z} = 1 - p_\mathrm{n}$.

The probability that a single spell hits the cat is equal to the ratio of the area of the cat's cross-section to the total area that can be hit. Thus,

$$p_\mathrm{z} = \frac{\pi r^2}{ab}\,.$$

The probability that a single spell does not hit the cat is then

$$p_\mathrm{n} = 1 - \frac{\pi r^2}{ab}\,.$$

For the cat to be dead, it must not be hit, not even once, and must be dead initially. Then

$$p_\theta = p_0 \left(1 - \frac{\pi r^2}{ab}\right)^3 ,$$

and the probability that the cat will be alive is calculated as

$$p_{\neg\theta} = 1 - p_0 \left(1 - \frac{\pi r^2}{ab}\right)^3 \doteq 64\,\%\,.$$

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Problem BF ... sphere in a cylinder

Consider a solid homogeneous sphere of radius r and mass m rolling inside a cylindrical cavity of inner radius $R = 3r$. The cavity is oriented so that its main axis is parallel to the horizontal plane, and the gravitational acceleration g acts vertically. Energy losses can be neglected for a few rotations, and the sphere rolls without slipping. The ratio of the sphere's speeds at the lowest point of the motion v_1 to the highest point v_2 is $v_1 = 7v_2/4 = 1.75v_2$. What is the maximum speed $v_{\max}$ reached by the sphere during its motion? Provide the result as a formula expressed only in terms of the parameters m, g, and r.

Karel wanted to include a picture with the problem, but he was too lazy to draw it.

The sphere inside the cylinder undergoes both translational and rotational motion, which are coupled because the sphere rolls without slipping. This can be expressed by the relation between its speed v, radius r, and angular velocity ω

$$v = r\omega\,.$$

The total kinetic energy of the sphere is the sum of translational and rotational energy:

$$E_i = \frac{1}{2}mv_i^2 + \frac{1}{2}J\omega_i^2\,,$$

where the index i denotes the situation 1 corresponding to the lowest point of motion and 2 to the highest point. The moment of inertia J for a solid homogeneous sphere is

$$J = \frac{2}{5}mr^2\,.$$

Thus, the kinetic energy can be rewritten as

$$E_i = \frac{7}{10} m v_i^2 \,.$$

The maximum speed occurs at the lowest point of the trajectory (v_1), and the difference in kinetic energy between the highest and lowest points is given by the difference in potential energies $\Delta E_\mathrm{p} = mg(h_2 - h_1)$. The lowest position of the sphere's center of mass is at height $h_1 = r$ above the bottom of the cylinder, and the highest position is at $h_2 = 2R - r = 6r - r = 5r$. Applying the law of conservation of energy yields

$$\begin{aligned}
\frac{7}{10} m v_1^2 &= \frac{7}{10} m v_2^2 + mg(h_2 - h_1)\,,\\
v_1^2 - v_2^2 &= \frac{10}{7} g(5r - r)\,,\\
v_1^2 - \frac{4^2}{7^2} v_1^2 &= \frac{10}{7} g \cdot 4r\,,\\
v_1 &= \sqrt{\frac{280}{33} gr} \approx 2.91\sqrt{gr} \approx \sqrt{\frac{r}{\mathrm{m}}} \cdot 9.12\,\mathrm{m{\cdot}s^{-1}}\,.
\end{aligned}$$

The maximum speed during the motion is thus $v_\mathrm{max} = v_1 = \sqrt{280gr/33}$ and is independent of m.

It remains to check that the problem is not a trick, that is, the sphere indeed completes the full rotation and does not fall. It would fall if the gravitational force exceeded the centripetal force required for the rotation. Comparing the centripetal acceleration to gravitational acceleration

$$a_\mathrm{cp} = \frac{v_2^2}{2r} = \frac{320}{231} g > g\,,$$

where we used $R - r = 2r$ for the radius of rotation, as this is the path the sphere follows. We see that gravity is not strong enough to pull the sphere down at the highest point, and therefore not at any other point along the trajectory.

In conclusion, we note that the constants were chosen close to the limit, but the sphere indeed remains on its path, and the problem was not intended as a trick.

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Problem BG ... a ball in hand

We would like to grab a large ball with one hand and hold it underneath. Let us model the situation as follows. We have $N = 5$ fingers, so we can act at N points. The friction between a finger and the ball is $f = 0.53$. The fingers are arranged symmetrically around the pole of the ball, which points upward in our gravitational field. What is the smallest zenith angle we must choose (that is, how close to the pole) so that we can still hold the ball? The mass of the ball is $550\,\mathrm{g}$, and the normal force that one finger can exert on the ball is $5.6\,\mathrm{N}$.

Jarda never understood how someone could hold a basketball this way.

To hold the ball at rest in the hand, we require that the resultant force acting on it be zero. Acting downward on it is the gravitational force mg, which must be compensated for by the friction force. For each finger, this force has magnitude fF, where F is the force that the finger exerts perpendicular to the surface of the ball.

Let this force make an angle α with the vertical plane. Then the friction force also makes an angle α with the horizontal plane and acts against the direction of possible motion of the ball, that is, toward the pole of the ball (the highest point). The horizontal component of the friction force is $Ff\cos\alpha$, and the vertical component, acting upward, is $Ff\sin\alpha$.

Now it is important to realize that in the horizontal direction, the components of the friction force Ff and of the normal force F cancel each other out, because we have distributed the fingers evenly and symmetrically around the vertical axis passing through the pole of the ball. In the vertical direction, the aforementioned gravitational force acts, as well as the friction force and also the vertical component of the normal force $F\cos\alpha$, which acts downward. For the resultant to be zero, the following must hold:

$$mg + NF\cos\alpha = NFf\sin\alpha\,.$$

We express $\sin\alpha$ in terms of the cosine as $\sqrt{1-\cos^2\alpha}$, substitute, and rearrange:

$$\begin{aligned} mg + NF\cos\alpha &= NFf\sqrt{1-\cos^2\alpha}\,,\\ m^2g^2 + 2mgNF\cos\alpha + N^2F^2\cos^2\alpha &= N^2F^2f^2 - N^2F^2f^2\cos^2\alpha\,,\\ m^2g^2 - N^2F^2f^2 + 2mgNF\cos\alpha + N^2F^2\left(f^2+1\right)\cos^2\alpha &= 0\,, \end{aligned}$$

from which, by solving the quadratic equation (we take the positive solution so that $\alpha < 90^\circ$), we obtain

$$\begin{aligned} \cos\alpha &= \frac{1}{2N^2F^2\left(f^2+1\right)}\left(-2mgNF + \sqrt{4m^2g^2N^2F^2 - 4\left(m^2g^2 - N^2F^2f^2\right)N^2F^2\left(f^2+1\right)}\right),\\ \cos\alpha &= \frac{1}{NF\left(f^2+1\right)}\left(-mg + f\sqrt{N^2F^2\left(f^2+1\right) - m^2g^2}\right),\\ \cos\alpha &= f\sqrt{\frac{1}{f^2+1} - \frac{m^2g^2}{N^2F^2\left(f^2+1\right)^2}} - \frac{mg}{NF\left(f^2+1\right)}\,,\\ \alpha &\doteq 72^\circ\,. \end{aligned}$$

If the angle were larger, the right side of the first equation would be larger than the left side, and we would easily hold the ball. In the opposite case, we would not hold the ball. By the equality, we have thus found the sought limiting value.

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Problem BH ... nuts and bolts

David once woke up with a strange dream: he wanted to fit a bolt with outer diameter $d_2 = 18.0\,\mathrm{mm}$ *into a nut with inner diameter* $d_1 = 12.0\,\mathrm{mm}$, *both measured at room temperature* $t_{\mathrm{room}} = 20.0\,^\circ\mathrm{C}$. *To achieve this, he bought a carafe of liquid nitrogen at a temperature of* $T_{\mathrm{N_2}} = 77.0\,\mathrm{K}$ *and placed the bolt inside. To what thermodynamic temperature must he heat the nut so that they can be screwed together? Neglect any expansion or contraction of the threads. The thermal expansion coefficient of both objects is* $\alpha = 345\cdot 10^{-6}\,\mathrm{K^{-1}}$, *and we assume it is independent of temperature.* *David told Matyáš about his big dream.*

We want the thermally expanded diameter of the nut to be equal to the thermally contracted diameter of the bolt. The formula for thermal expansion is

$$l_T = l_0(1 + \alpha\Delta T)\,.$$

Since the dimensions must match, we know that

$$d_1(1 + \alpha(T_{\mathrm{nut}} - T_{\mathrm{room}})) = d_2(1 + \alpha(T_{\mathrm{N_2}} - T_{\mathrm{room}}))\,,$$

where, after rearranging, we obtain

$$T_{\mathrm{nut}} - T_{\mathrm{room}} = \frac{d_2(1 + \alpha(T_{\mathrm{N_2}} - T_{\mathrm{room}}))}{d_1\alpha} - \frac{1}{\alpha}$$

and finally, we solve for T_{nut}:

$$T_{\mathrm{nut}} = \frac{d_2(1 + \alpha(T_{\mathrm{N_2}} - T_{\mathrm{room}})) - d_1}{d_1\alpha} + T_{\mathrm{room}} \doteq 1\,420\,\mathrm{K}\,.$$

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Problem CA ... as quick as snapping your fingers

In the film Avengers: Infinity War*, the villain Thanos turns half of the living beings in the universe into dust with a snap of his fingers. Suppose he achieves this by making all atoms in the bodies of these organisms unstable with a very short half-life. Within $T = 5\,\mathrm{s}$ after the snap, the affected beings decay by $p = 99\%$. What was their half-life?*

The other half of the universe died of cancer from the accompanying radiation.

We use the knowledge of the radioactive decay formula,

$$N(t) = N_0 e^{-\lambda t}\,,$$

which yields the number of undecayed particles N at time t, if N_0 is the initial number of particles. The decay constant λ is related to the half-life $T_{1/2}$ by

$$\lambda = \frac{\ln 2}{T_{1/2}}\,.$$

At the given time T, the number of decayed particles satisfies the equation

$$p = \frac{N_0 - N(T)}{N_0} = 1 - e^{-\frac{\ln 2}{T_{1/2}}T}\,.$$

It is now sufficient to solve for the half-life $T_{1/2}$, leading to a result

$$T_{1/2} = \frac{\ln 2}{\ln\frac{1}{1-p}}T \doteq 0.75\,\mathrm{s}\,.$$

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Problem CB ... how a wheel went wandering

Paťo was driving a car uphill along a long, straight, empty road with a constant inclination angle of $\alpha = 3.8^\circ$. Suddenly, he heard a strange sound, looked into the rearview mirror, and his heart sank: the rear wheel of his car was rolling behind the vehicle. The car, however, continued on as if nothing had happened, so he started thinking about what to do with the wheel. Trying to stop it manually would be dangerous, so he decided to wait for it at the place where it would come to rest. At what distance from the point of detachment does the wheel stop?

Consider the wheel to be a rigid homogeneous cylinder of mass $m = 21\,\mathrm{kg}$ and radius $r = 32\,\mathrm{cm}$. Immediately after detachment, the wheel had a tangential speed $v = 90\,\mathrm{km{\cdot}h^{-1}}$ and rolled without slipping along the car's straight path until it came to rest. At the same time, the wheel is decelerated opposite to the direction of motion by a resistive force of magnitude kF_N acting at the center of mass, where $k = 2.4 \cdot 10^{-2}$ is a proportionality coefficient and F_N is the normal force exerted by the road on the wheel. Neglect any deformation of the wheel.

Driving seems dangerous to Paťo.

Solution using work and energy

After detaching, the wheel moved with a nonzero speed v; for this reason, we calculate its initial kinetic energy E_k. This is given by the sum of the translational kinetic energy

$$E_\mathrm{k,t} = \frac{1}{2}mv^2$$

representing the translational motion of the wheel's center of mass; and the rotational kinetic energy $E_\mathrm{k,r}$ describing its rotational motion, for which

$$E_\mathrm{k,r} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\frac{v^2}{r^2} = \frac{1}{4}mv^2$$

holds. In the previous relation, $I = mr^2/2$ represents the moment of inertia of a solid homogeneous cylinder about its axis, and ω denotes the angular velocity of the wheel, for which, due to the no-slip condition, $v = \omega r$ holds.

Without loss of generality, we may choose the zero level of potential energy (of the gravitational field) at the point where the wheel detaches. A resistive force also acts on the wheel; however, at the moment of detachment, it has done no work yet, since the wheel has not yet traveled any distance. The total initial energy E_1 of the wheel immediately after detachment is then

$$E_1 = E_\mathrm{k} = E_\mathrm{k,t} + E_\mathrm{k,r} = \frac{3}{4}mv^2\,.$$

Let us now consider the final state of the wheel when it comes to rest. Its kinetic energy must necessarily be zero; however, as it moves uphill, the potential energy increases. With respect to the chosen zero level at the point of detachment, the final potential energy E_p is given by

$$E_\mathrm{p} = mg\Delta h = mgs\sin\alpha\,,$$

where $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is the standard gravitational acceleration and Δh is the height difference. We determined this difference from the geometry of the situation using the incline angle α and the traveled distance s, which we seek. At the same time, the deceleration is also contributed to by the resistive force kF_N, where the normal force satisfies $F_\mathrm{N} = mg\cos\alpha$. The resistance is constant throughout the entire process, and the work W of the resistive force over the distance s is then

$$W = kF_\mathrm{N}s = kmgs\cos\alpha\,.$$

From the law of conservation of energy, it follows that the entire initial energy E_1 must be converted into an increase of potential energy E_p and into heat given by the work W of the resistive force:

$$E_1 = E_\mathrm{p} + W\,,$$

$$\frac{3}{4}mv^2 = mgs\sin\alpha + kmgs\cos\alpha\,.$$

From this, we simply solve for the distance s and, by substituting numerical values, evaluate the distance over which the wheel comes to rest:

$$s = \frac{3v^2}{4g\left(\sin\alpha + k\cos\alpha\right)} \doteq 0.53\,\mathrm{km}\,. \tag{1}$$

Solution using forces

Alternatively, the problem can be solved by setting up the force and torque equations of motion. The resultant of the external forces will decelerate the translational motion of the center of mass, but at the same time their torques will begin to slow the rotation of the entire wheel. Thus, these two processes are coupled by the no-slip condition (in this case $a = \varepsilon r$ for the translational acceleration a of the center of mass and the angular acceleration ε of the entire wheel), which is ensured by the static friction force F_t.

In addition to the friction force F_t, the wheel is acted upon by the component of the gravitational force $mg\sin\alpha$ directed opposite to the motion of the wheel, as well as by the resistive force $kF_\mathrm{N} = kmg\cos\alpha$ of the same direction. When rolling forward, the wheel tends to slip backward on the surface. Therefore, the friction force F_t then acts in the direction of motion, exactly opposite to the other forces. If we choose the current direction of rolling of the wheel as positive, the force equation for the translational motion has the form

$$ma = -mg\sin\alpha - kmg\cos\alpha + F_\mathrm{t}\,. \tag{2}$$

Similarly, for the rotation of the wheel about its axis, we can set up the torque equation. Both the gravitational and the resistive forces act at the center of mass lying on the axis, so their torque is zero. The friction force F_t acts in the plane of the road at the point of contact with the wheel. Thus, its torque with respect to the wheel axis is, due to the perpendicularity of the tangent at the point of contact to the radius of the circle (or cylinder), equal to $F_\mathrm{t}r$. We obtain the torque equation

$$I\varepsilon = -F_\mathrm{t}r\,.$$

The negative sign of the torque of the friction force follows from the fact that, although the friction force has the direction of motion of the wheel, it causes rotation in exactly the opposite (negative) direction. After substituting $I = mr^2/2$ and $a = \varepsilon r$, we express the friction force F_t as

$$F_\mathrm{t} = -\frac{1}{2}ma\,.$$

Substituting into the equation of motion (2), we obtain the acceleration

$$a = -\frac{2}{3} g \left(\sin\alpha + k\cos\alpha\right) .$$

Since the acceleration is constant, we may apply the standard kinematic equations for uniformly accelerated motion. For the velocity $u(t)$ of the wheel as a function of time (in the positive direction), we have

$$u(t) = at + v \; = -\frac{2}{3} gt \left(\sin\alpha + k\cos\alpha\right) + v\,.$$

At the time t_z when the wheel stops, the velocity is zero ($u(t_\mathrm{z}) = 0$). Thus, from the equation it can be expressed as

$$t_\mathrm{z} = \frac{3v}{2g\left(\sin\alpha + k\cos\alpha\right)}\,.$$

The sought distance s that the wheel travels with uniformly decelerated motion during the time t_z is then

$$s = \frac{1}{2} a t_\mathrm{z}^2 + v t_\mathrm{z} = \frac{1}{2}\left(-\frac{2}{3} g \left(\sin\alpha + k\cos\alpha\right)\right)\left(\frac{9v^2}{4g^2\left(\sin\alpha + k\cos\alpha\right)^2}\right) + \frac{3v^2}{2g\left(\sin\alpha + k\cos\alpha\right)} =$$

$$= -\frac{3v^2}{4g\left(\sin\alpha + k\cos\alpha\right)} + \frac{3v^2}{2g\left(\sin\alpha + k\cos\alpha\right)} = \frac{3v^2}{4g\left(\sin\alpha + k\cos\alpha\right)} \doteq 0.53\,\mathrm{km}\,,$$

which agrees with the result (1).

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Problem CC ... Wien filter

The Wien filter *is a device used to select from a beam of charged particles only those with a specific velocity. It consists of two parallel plates between which a uniform electric field of magnitude E is established, together with a uniform magnetic field of magnitude B that is oriented perpendicular to the electric field. The particle beam enters the region between the plates with its velocity perpendicular to both fields. For given values of E and B, at what velocity will the particles pass through the filter without any change in the direction of their motion?* *Petr was attending a lecture on nuclear physics.*

We need to express the electric and magnetic force

$$\mathbf{F}_E = q\mathbf{E}\,,$$
$$\mathbf{F}_B = q\left(\mathbf{v} \times \mathbf{B}\right) .$$

In a Wien filter, the directions of the electric and magnetic fields are chosen so that the forces on a passing particle act exactly in opposite directions. The problem can thus be treated one-dimensionally; since the particle velocity and **B** are perpendicular, the vector cross product reduces to a simple multiplication. For the particle to pass through the filter, its motion must not be deflected at all, hence

$$F_E - F_B = 0\,,$$
$$qE - qvB = 0\,.$$

This gives a simple condition for the particle velocity

$$v = \frac{E}{B}\,.$$

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Problem CD ... spinning sink

Imagine a sink in the shape of a hollow cylinder with a radius r. At one point along the bottom edge, water jets out and lands directly at the center of the sink's bottom. The water reaches a maximum height h above the surface. What is the maximum angular velocity at which we can spin the sink together with the nozzle about the cylinder axis so that the water does not hit the cylinder wall? *The FYKOS team visited CosmoCaixa.*

As long as the sink is not spinning, we have

$$r = v_0 t \cos\alpha\,,$$

where r is the sink radius, v_0 is the initial speed of the water (with respect to the sink and now also with respect to the ground), and $\cos\alpha$ is the initial direction with respect to the ground. The time t, during which the water drops are in the air, can be expressed from vertical velocity

$$0 = v_0 \sin\alpha - g\frac{t}{2}\,,$$
$$t = \frac{2v_0 \sin\alpha}{g}\,,$$

which is in turn related to the reached height h through the law of conservation of energy as

$$\frac{1}{2}m\left(v_0 \sin\alpha\right)^2 = mgh\,,$$
$$v_0^2 \sin^2\alpha = 2gh\,.$$

From the given parameters, we can therefore express

$$t = \sqrt{\frac{8h}{g}}\,.$$

When we spin the sink with angular speed ω, with respect to the ground, the water drops acquire an additional horizontal velocity component which is tangential to the motion and has magnitude $v_\perp = \omega r$.

We know that in the normal direction, the water travels a distance r. However, in order not to land at the center of the circle but at its edge, it must also travel a distance r in the tangential direction (this is the distance between the center and the edge in this direction). Thus, it must hold that

$$v_\perp t = r = \omega r t \quad \Rightarrow \quad \omega = \frac{1}{t} = \sqrt{\frac{g}{8h}}\,.$$

The sought angular speed is therefore

$$\omega = \sqrt{\frac{g}{8h}}\,.$$

If we were to spin the sink faster, the water would strike the cylinder wall at a nonzero height.

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Problem CE ... Titanic

The Titanic is sailing at a speed $v = 45.0\,\mathrm{km\cdot h^{-1}}$ toward an iceberg when the ship's captain sounds the horn, which emits a sound with frequency $f = 440\,\mathrm{Hz}$. After the horn falls silent, the sound reflects off the iceberg back toward the ship. What frequency f' does the captain hear?

If Peter had been the captain, he might have noticed it.

It should be noted that both the Titanic and the iceberg act as transmitter and receiver, depending on the direction. If the Titanic emits a sound with frequency f, this means that the Titanic is a source moving at speed v and the iceberg is a stationary receiver, which detects the sound at frequency

$$f_\mathrm{i} = f\frac{c_\mathrm{s}}{c_\mathrm{s} - v}\,.$$

The sound reflects off the iceberg still at frequency f_i, so the iceberg behaves as a stationary source. The Titanic is now moving toward the sound, acting as a moving receiver, and therefore detects the sound at frequency

$$f' = f_\mathrm{i}\frac{c_\mathrm{s} + v}{c_\mathrm{s}} = f\frac{c_\mathrm{s}}{c_\mathrm{s} - v}\frac{c_\mathrm{s} + v}{c_\mathrm{s}} = f\frac{c_\mathrm{s} + v}{c_\mathrm{s} - v} \doteq 473\,\mathrm{Hz}\,.$$

In other words, the iceberg acts like a mirror, so it is as if two Titanics are approaching each other, the first as source and the second as receiver.

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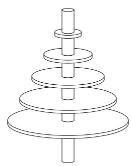
Problem CF ... Christmas tree



Martin wanted to get rid of the needles from a Christmas tree by spinning it about its axis of symmetry with an angular speed $\omega = 5.5\,\mathrm{rad\cdot s^{-1}}$. Let us approximate the Christmas tree as a rod of height $h = 1.5\,\mathrm{m}$ and radius $r/2$, and five disks with radii $1r$, $2r$, $3r$, $4r$, $5r$, where $r = 15\,\mathrm{cm}$, with thickness $l = 5.0\,\mathrm{mm}$. The central rod passes through the disks (meaning that they have holes in the center) and has density $\rho = 900\,\mathrm{kg\cdot m^{-3}}$. The disks have half this density. Martin is interested in how much energy is required to achieve angular speed ω.

The organizers and Martin were waiting for a lecture.

The kinetic energy of rotational motion is $E_\mathrm{k} = I\omega^2/2$, so we only need to determine the moment of inertia of the tree I.

Let us ignore the holes in the disks for the moment. The moment of inertia of a solid disk with mass m and radius R is $I = mR^2/2$. We compute the mass as the product of density and volume, so in this case $m = (\rho/2)\pi R^2 l$. Altogether, for a disk of radius R, we obtain the moment of inertia

$$I(R) = \frac{1}{4}\pi\rho l R^4\,.$$

Substituting successively $R = ir$, we obtain the total moment of inertia of the disks

$$I_\mathrm{d} = \sum_{i=1}^{5} \frac{1}{4}\pi\rho l (ir)^4 = \frac{1}{4}\pi\rho l r^4 (1 + 16 + 81 + 256 + 625) = \frac{979}{4}\pi\rho l r^4 \doteq 1.75\,\mathrm{kg{\cdot}m^2}\,.$$

We are left with the central rod. It also has the shape of a disk, but with a relatively large height compared to its radius. The rod itself would have mass $\rho\pi(r/2)^2 h$, but in the regions where it passes through the disks, we have already included half of its density in the disks. We therefore have to subtract $(\rho/2)\pi(r/2)^2 5l$. Its resulting moment of inertia is thus

$$I_\mathrm{p} = \frac{1}{2}\pi\rho\frac{r^2}{4}\left(h - \frac{5}{2}l\right)\frac{r^2}{4} = \frac{1}{32}\pi\rho\left(h - \frac{5}{2}l\right) r^4 \doteq 0.07\,\mathrm{kg{\cdot}m^2}\,.$$

We see that $I_\mathrm{p} \ll I_\mathrm{d} \Rightarrow I_\mathrm{p} + I_\mathrm{d} \approx I_\mathrm{d}$, which may not be particularly surprising. In any case, within our accuracy, such an approximation would not be sufficient, so we compute the required energy as

$$E_\mathrm{k} = \frac{1}{2}(I_\mathrm{p} + I_\mathrm{d})\omega^2 \doteq 27.5\,\mathrm{J}\,.$$

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Problem CG ... table oscillation

A body of mass $m = 100\,\mathrm{g}$ is placed on a plate that undergoes harmonic oscillations in its plane with angular frequency ω and amplitude $A = 3.0\,\mathrm{cm}$. What is the limiting value of ω such that the plate begins to slip under the body? The coefficient of friction between the body and the plate is $f = 0.60$. *Pepa tutored mechanics.*

The displacement of the plate from its equilibrium position during its oscillatory motion can be expressed by the standard relation

$$x = A\sin(\omega t)\,.$$

During such motion of the plate, an inertial force of magnitude $|F| = m\ddot{x}$ acts on the body in the direction opposite to the acceleration $\ddot{x}$ of the plate. The plate begins to slip at the moment when the magnitude of this force exceeds the maximum friction force. In this limiting case, the relation holds

$$m\ddot{x} = fmg\,.$$

The acceleration of the plate can be determined using a well-known relation for acceleration in harmonic motion, $\ddot{x} = -\omega^2 x$, or it can be derived as the second time derivative of the displacement,

$$\ddot{x} = -A\omega^2\sin(\omega t) = -\omega^2 x\,.$$

The acceleration's sign only indicates its direction and can be ignored. The magnitude of the acceleration of the plate then increases with the magnitude of its displacement; therefore, the maximum magnitude of this acceleration is

$$\ddot{x}_{\max} = \omega^2 A\,.$$

It thus suffices to focus on the case in which the inertial force at the moment of maximum acceleration just exceeds the value of the friction force. This limiting case occurs for

$$m\omega^2 A = fmg\,,$$

from which we can, by simple algebraic manipulation, express the required minimum angular frequency as

$$\omega = \sqrt{\frac{fg}{A}} \doteq 14\,\mathrm{rad{\cdot}s^{-1}}\,.$$

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Problem CH ... Plato's current

What is the electric flux through one face of a regular icosahedron (twenty-sided polyhedron) with charge of magnitude Q located in its center?

Jarda was rolling D20 dice.

The total flux of the electric field through a closed surface is, according to Gauss's law,

$$\oint \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0}\,.$$

Since all 20 faces of the icosahedron are identical, the flux through one face is $Q/(20\varepsilon_0)$, which is the solution to our problem.

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Problem DA ... warming rezistor

Consider a voltage source $U = 250\,\mathrm{V}$ and a resistor whose resistance varies with temperature according to $R(T) = R_0(1 + \alpha\Delta T)$, where $R_0 = 5.0\,\Omega$ is the resistance at room temperature, $\alpha = 4.9\cdot 10^{-3}\,\mathrm{K^{-1}}$ is the temperature coefficient of resistance, and ΔT is the difference between the resistor's temperature and room temperature. Assume that the resistor's temperature is higher than the surroundings by $\Delta T = \beta P$, where P is the power dissipated in the resistor and $\beta = 1.5\cdot 10^{-2}\,\mathrm{K{\cdot}W^{-1}}$. What is the steady-state current?

Lego built an electric circuit.

The power dissipated in the resistor is $P = UI = U^2/R$. Substituting this into the formula for the temperature difference, and subsequently into the formula for resistance, we obtain

$$R = R_0 + R_0\alpha\beta\frac{U^2}{R}\,.$$

This can be rearranged into a quadratic equation for the resistance:

$$R^2 - RR_0 - R_0\alpha\beta U^2 = 0\,.$$

The solutions of this equation are

$$R_{1,2} = \frac{R_0 \pm \sqrt{R_0^2 + 4R_0\alpha\beta U^2}}{2}\,.$$

The solution with the minus sign yields a negative resistance, which is physically meaningless and corresponds to an “unstable equilibrium”. Therefore, we take the solution with the plus sign. The steady-state current is then

$$I = \frac{U}{R} = \frac{2U}{R_0 + \sqrt{R_0^2 + 4R_0\alpha\beta U^2}} = \frac{\sqrt{R_0^2 + 4R_0\alpha\beta U^2} - R_0}{2R_0\alpha\beta U} \doteq 32\,\mathrm{A}\,.$$

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Problem DB ... triple collision

Two identical smooth spheres with radii $r = 10\,\mathrm{cm}$ lie at rest on a horizontal table with their centers separated by a distance $d = 30\,\mathrm{cm}$. A third identical sphere approaches from a distance with velocity $v = 1.0\,\mathrm{m\cdot s^{-1}}$ along the perpendicular bisector of the segment connecting their centers. All collisions are instantaneous and perfectly elastic. What will be the velocity of the incoming sphere after the collisions? Submit a positive result if it moves in its original direction, and a negative one if it moves in the opposite direction.

Lego wanted to create a problem in Jarda’s style.

Since the spheres are perfectly smooth, there is no friction between them; consequently, they will exert only normal forces on each other during the collision. This implies that each initially stationary sphere will move off in the direction given by the line connecting its center to the center of the incoming sphere. We determine this direction from a right triangle whose hypotenuse is the connecting line (length $2r$) and one leg is half the segment between the stationary spheres (length $d/2$). Thus, the direction of motion will make an angle φ with the direction of the incoming sphere given by

$$\varphi = \arcsin\frac{d}{4r}\,.$$

Furthermore, it is clear from symmetry that after the collision, the incoming sphere will continue to move either in the direction of its initial velocity or in the exact opposite direction. Let us denote this velocity as v_1 (taking the direction of its initial velocity as positive). Symmetry also implies that the two other spheres will have equal speed; let us denote this speed as v_2.

Then, the law of conservation of energy implies

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + 2\cdot\frac{1}{2}mv_2^2\,,$$
$$v^2 = v_1^2 + 2v_2^2\,.$$

And from the law of conservation of momentum

$$mv = mv_1 + 2mv_2 \cos\varphi\,,$$
$$v = v_1 + 2v_2 \cos\varphi\,,$$

since the momentum components in the perpendicular direction cancel out. We express $v_1 = v - 2v_2 \cos\varphi$ from the law of conservation of momentum and substitute it into the law of conservation of energy equation

$$\begin{aligned} v^2 &= (v - 2v_2\cos\varphi)^2 + 2v_2^2\,,\\ v^2 &= v^2 - 4vv_2\cos\varphi + 4v_2^2\cos^2\varphi + 2v_2^2\,,\\ 4v\cos\varphi &= 4v_2\cos^2\varphi + 2v_2\,,\\ \frac{2v\cos\varphi}{2\cos^2\varphi + 1} &= v_2\,, \end{aligned}$$

where we discarded the solution $v_2 = 0$, as this would correspond to a situation where no collision occurs. The final step is to substitute back into the law of conservation of momentum, obtaining

$$v_1 = v - 2v_2\cos\varphi = v - \frac{4\cos^2\varphi}{2\cos^2\varphi + 1}v = \frac{1 - 2\cos^2\varphi}{2\cos^2\varphi + 1}v\,.$$

Trigonometric identities yield $\cos(\arcsin x) = \sqrt{1 - x^2}$, which means

$$v_1 = \frac{d^2 - 8r^2}{24r^2 - d^2}v \doteq 0.067\,\mathrm{m\cdot s^{-1}}\,.$$

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Problem DC ... Watt's magnetic governor

Consider a classic Watt governor consisting of a vertical axis to which two massless arms of length $l = 30.0\,\mathrm{cm}$ are freely attached at a single common joint. Small spherical weights of mass $m = 100\,\mathrm{g}$ are attached to the ends of the arms. When the axis starts spinning, the arms begin to rise due to the centrifugal force. In our situation, the weights are additionally charged with identical charges $q = 2.00\,\mathrm{\mu C}$, and the entire system is placed in a homogeneous magnetic field of hypothetical magnitude $B = 750\,\mathrm{kT}$ oriented along the axis. At what minimum angular speed ω can the arms open to an angle of $2\vartheta = 90.0^\circ$? *Peter is into electromagnetism.*

We break down the force acting on a weight into its individual components. In the vertical direction, only the force of gravity acts on the weight:

$$F_z = mg\,,$$

where we have chosen the positive direction of the axis to point downward. In the radial direction, we must account for the centrifugal, electrostatic, and magnetic force; for these, respectively, we have

$$F_{\mathrm{od}} = m\omega^2 r\,,$$
$$F_{\mathrm{el}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2r)^2}\,,$$
$$F_{\mathrm{mag}} = \pm q\omega r B\,,$$

where r is the distance of the weight from the axis. In deriving the relation for F_{mag}, we used the expression for the magnetic force

$$\mathbf{F} = q\left(\mathbf{v}\times\mathbf{B}\right)\,,$$

into which we substituted the relation for speed during circular motion $v = \omega r$, and then realized that the motion takes place in a horizontal plane perpendicular to the magnetic field; the vector product therefore reduces to an ordinary product, and the resulting vector points in the radial direction. We must also take into account that we do not know the direction of the magnetic induction B nor the direction in which the governor rotates. Therefore, we also do not know the direction of the magnetic force (although from the requirement that we are looking for the minimum angular speed, we suspect that the governor rotates in such a direction that the resulting force points radially away from the rotation axis). From geometry, we find that $r = l\sin\vartheta$, and therefore the total radial force is

$$F_{\mathrm{r}} = m\omega^2 l\sin\vartheta + \frac{q^2}{4\pi\varepsilon_0}\frac{1}{4l^2\sin^2\vartheta} \pm q\omega l B\sin\vartheta\,.$$

For the system to be in equilibrium, the direction of the resultant force must be parallel to the arm to which the weight is attached. It must therefore hold that

$$\tan\vartheta = \frac{F_{\mathrm{r}}}{F_z}\,,$$

which, after rearrangement, leads to a quadratic equation

$$\omega^2 \pm \frac{qB}{m}\omega + \frac{q^2}{16\pi l^3 m\varepsilon_0}\frac{1}{\sin^3\vartheta} - \frac{g}{l}\frac{1}{\cos\vartheta} = 0\,,$$

whose roots are

$$\omega = \begin{cases} \mp\left(\frac{qB}{2m} - \frac{1}{2}\sqrt{\frac{q^2B^2}{m^2} - \frac{q^2}{4\pi l^3 m\varepsilon_0}\frac{1}{\sin^3\vartheta} + \frac{4g}{l}\frac{1}{\cos\vartheta}}\right) = \pm 2.15\,\mathrm{rad\cdot s^{-1}}\,,\\ \mp\left(\frac{qB}{2m} + \frac{1}{2}\sqrt{\frac{q^2B^2}{m^2} - \frac{q^2}{4\pi l^3 m\varepsilon_0}\frac{1}{\sin^3\vartheta} + \frac{4g}{l}\frac{1}{\cos\vartheta}}\right) = \mp 17.1\,\mathrm{rad\cdot s^{-1}}\,. \end{cases}$$

The problem statement asked for the *smallest* magnitude of the angular velocity at which the opening occurs; the correct solution is therefore the smaller magnitude

$$|\omega| = 2.15\,\mathrm{rad\cdot s^{-1}}\,.$$

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Problem DD ... overturned train

Grandmother learned about the Coriolis force, and while traveling by train from Prague exactly due south, she panicked, thinking that the train cars must not travel too fast so that they would not tip over. The train car has a rectangular cross section of width $a = 3\,150\,\mathrm{mm}$ *and height* $b = 4\,320\,\mathrm{mm}$, *with its center of mass located at the center of the cross section. Prague lies at approximately* 50° *north latitude. At what minimum speed would the car have to travel? An estimate is sufficient; relativistic effects can be neglected. Give the result to two significant figures.*

Hint: *In a reference frame rotating with angular velocity* $\boldsymbol{\omega}$, *the Coriolis force acting on a body of mass* m *moving with velocity* $\mathbf{v}$ *is given by* $\mathbf{F}_{\mathrm{Cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$.

Petr sat in a train where strange things happened on trains.

If we are interested only in the magnitude of the Coriolis force, it suffices, in the relation given in the hint, to replace the vector product by the simple product of the magnitudes ω and v and the sine of the angle between them. The vector $\boldsymbol{\omega}$ points along the axis of rotation; however, geographic latitude is measured from the equator. Let us denote the latitude angle by

$$\vartheta = 50^\circ\,,$$

then we can see that the angle between the vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ is $\pi - \vartheta$, which, after adjusting the sine and neglecting the sign that only indicates the orientation of the force, gives the magnitude of the Coriolis force

$$F_{\mathrm{Cor}} = 2m\omega v \sin\vartheta\,.$$

For the car to tip over, the torque by which the Coriolis force acts on it must balance the torque by which the gravitational force acts on it. It is easy to see that the gravitational force produces its maximum torque at the moment when the car is not tilted at all. This means that it suffices to overcome the gravitational force only at the onset of tipping; afterward, its influence will always be smaller. If we denote by α the angle between the diagonal of the car's cross section and the vertical axis, the following must hold:

$$mg\sin\alpha = 2m\omega v \sin\vartheta\cos\alpha\,.$$

If we rearrange this relation, use $\tan\alpha = a/b$, and express $\omega = 2\pi/T$, where T is the period of the Earth's rotation (that is, approximately $24\,\mathrm{h}$), we obtain

$$v = \frac{gT}{4\pi}\frac{a}{b}\frac{1}{\sin\vartheta} \doteq 64\,\mathrm{km{\cdot}s^{-1}}\,.$$

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Problem DE ... Where to with it?

Jan Neruda is fed up with his old straw mattress. Instead of gradually scattering it from his trouser legs during walks, he decides to bundle the mattress into a small ball of mass $m = 20\,\mathrm{kg}$ *and launch it into the Vltava River using a homemade catapult from his house at Konviktská Street No. 30. The river is at a distance* $L = 250\,\mathrm{m}$, *and the middle* $13/15$ *of the trajectory is obstructed by houses of height* $h = 18\,\mathrm{m}$, *which he must shoot over. The final step is to choose* *a sufficiently stiff spring for the catapult to power the launch. What is the minimum spring constant required if the catapult allows a maximum extension of $x = 1.25\,\mathrm{m}$? The catapult can be fired at any angle.* *Petr read Neruda's famous feuilleton "Kam s ním?".*

Let us consider the parabolic trajectory along which the straw mattress will fly. If we place the origin of our coordinate system at the midpoint of the distance it must cover, we can write the equation of the trajectory in the form

$$y = -al^2 + H\,,$$

where l is the horizontal distance from the midpoint of the trajectory, H is the maximum height reached by the straw mattress, and a is a parameter determining the shape of the parabola. Because the straw mattress is on the ground at the beginning, we have

$$0 = -a\left(\frac{L}{2}\right)^2 + H\,.$$

For the straw mattress to clear the row of houses in its path while reaching the smallest possible height H, it must hold that

$$h = -a\left(\frac{13}{30}L\right)^2 + H\,.$$

We therefore obtain the system of equations

$$a = \frac{225h}{14L^2} \doteq 4.63 \cdot 10^{-3}\,\mathrm{m}^{-1}\,,$$
$$H = \frac{225h}{56} \doteq 72.32\,\mathrm{m}\,.$$

For the straw mattress to reach the height H, its initial vertical velocity must be such that (due to the law of conservation of energy)

$$\frac{1}{2}mv_y^2 = mgH \quad\Rightarrow\quad v_y = \sqrt{2gH}\,.$$

At the same time, however, the straw mattress must initially have a suitably large horizontal component of velocity; if it has an incorrect value, it will follow a trajectory different from the one we desire. The correct horizontal velocity is such that the total velocity is tangent to the trajectory. We know that the tangent of the tangent line to a graph at a given point equals the derivative at that point. For our derivative in general, and specifically at the beginning of the trajectory, we have

$$y' = -2al \quad\Rightarrow\quad y'\left(-\frac{L}{2}\right) = aL\,,$$

Using the property of the derivative mentioned above, we obtain

$$\frac{v_y}{v_x} = aL \quad\Rightarrow\quad v_x = \frac{\sqrt{2gH}}{aL}\,.$$

Let us pause here to consider one more idea: if it were the case that

$$\left|\frac{v_y}{v_x}\right| < 1 = \tan\frac{\pi}{4}\,,$$

that is, if a shot that just barely cleared the houses in its path were launched at an angle smaller than 45°, it would be more advantageous to launch the straw mattress simply at an angle of 45°, because at this angle the ratio of range to required energy is optimal. In our case, however, $|aL| \doteq 1.16$ holds, which means that we want to launch the bundle at an angle for which equality holds between the ratio of v_x, v_y, and aL.

For the catapult to achieve the required range, it must be possible to "store" its entire initial kinetic energy in the stretched spring. Using the expression for the potential energy of a spring, we therefore have

$$\frac{1}{2}m\left(v_x^2+v_y^2\right)=\frac{1}{2}kx^2$$

and from this, by expressing the spring constant k and substituting, we obtain

$$k=\frac{225}{28}\frac{mgh}{x^2}\left(\frac{1+a^2L^2}{a^2L^2}\right)\doteq 32\,\mathrm{kN\cdot m^{-1}}\,.$$

The problem is inspired by "Kam s ním—where to with it", a famous column by Jan Neruda, in which the author discusses how to get rid of an old straw mattress, at a time when there was nowhere to throw it away. The author suggests, for example, gradually scattering the straw from one's trouser legs.

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Problem DF ... hellish

In the deepest abyss of Hell, in the very ninth circle, Lucifer himself with three faces is embedded in the middle of a frozen surface, holding a traitor in each mouth. Around him, a pentagram is drawn on the ice, consisting of a circle of radius R in which a regular five-pointed star is inscribed. A charge Q is located at three of the star's vertices, and a charge q of the same sign is located at three of the star's intersections. What must be the ratio Q/q so that the electric field at the center of the pentagram, where Lucifer is located, is zero?

Peter read Divine comedy.

From the problem statement, we know that the charges Q are at a distance R from the center of the pentagram. However, we also need to determine the distance of the charges q. They lie on a smaller circle; let us denote its radius as a. The intersections of the pentagram form a regular pentagon. Dividing it into five triangles, we find that each triangle has a vertex angle $\alpha = 360°/5 = 72°$ at the center of the circle, and the angle at the remaining two vertices is $\beta = 54°$.

Consider the figure formed by one tip of the star and the adjacent triangle from the inner pentagon. The sum of the altitudes of these triangles is exactly R, and all interior angles of the figure can be determined from the known α and β. Using some trigonometry and algebra, we can express a as

$$a=\frac{R}{\tan 72°\cos 54°+\sin 54°}\,.$$

Let us consider how the charges are arranged. On one circle, the charges can be positioned either such that all three are adjacent, or such that two are adjacent and the third is opposite them. In any case, by symmetry, we can consider that on the second circle, the charges must be arranged as if the charges on the first circle were reflected through the center of the circle and scaled to the radius of the second circle.

Then, expressing the electric field at the center of the circle as the sum of the fields from the charges on the inner circle $\mathbf{E}_q$ and from the charges on the outer circle $\mathbf{E}_Q$, we require a magnitude of charges such that

$$\mathbf{E}_q + \mathbf{E}_Q = 0\,.$$

Due to symmetry, up to a sign, $\mathbf{E}_q$ and $\mathbf{E}_Q$ have the same form and differ only by the factor Q/R^2 and q/a^2, respectively. We thus obtain

$$\frac{Q}{R^2} = \frac{q}{a^2} \quad \Rightarrow \quad \frac{Q}{q} = \left(\frac{R}{a}\right)^2 = \left(\tan 72^\circ \cos 54^\circ + \sin 54^\circ\right)^2 \doteq 6.85\,.$$

It is also interesting to note that

$$\sqrt{\frac{R}{a}} = \varphi = 1.61803\ldots\,,$$

where φ is the golden ratio.

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Problem DG ... bead on parabole

Consider a bead of mass m sliding freely on a wire in the shape of a parabolic curve $y = ax^2$ in a uniform gravitational field. The wire is rotated about the axis ("the y-axis"), which is parallel to $\mathbf{g}$. What must be the angular velocity ω so that the bead does not slide along the parabola regardless of its position? *Petr reminisced about theoretical mechanics.*

Let us express the forces acting on the bead as a vector. In the x-direction, there is the centrifugal force, and in the y-direction, there is the gravitational force. Thus,

$$\mathbf{F} = \begin{bmatrix} m\omega^2 x \\ -mg \end{bmatrix}\,.$$

The bead can move only along the parabola. To remain stationary, the resultant force on the bead must act along the normal to the parabola; otherwise, the tangential component would cause the bead to "slide" along the parabola. The resultant force points along the normal precisely when it is perpendicular to the tangent to the parabola. The tangent can be expressed using the derivative. We have

$$y' = 2ax\,,$$

so a vector along the tangent at point x is

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2ax \end{bmatrix}\,.$$

Perpendicularity is verified by requiring that $\mathbf{v}$ and $\mathbf{F}$ have a zero scalar product. The condition on ω is therefore

$$\mathbf{F} \cdot \mathbf{v} = m\omega^2 x - 2mgax = 0\,.$$

From this, it follows that

$$\omega = \sqrt{2ga}\,.$$

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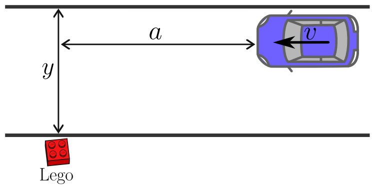
Problem DH ... crossing the road in front of a car

Lego sometimes cuts it a bit too close when running across the road, so he decided that this time he would rather calculate everything in advance, in case the car is driven by Radek, who does not slow down for pedestrians. The situation is shown in the figure. What is the minimum speed at which Lego must run to cross in front of the car? Lego does not necessarily have to run perpendicular to the road. Express the result in terms of v, y, a.



Before Lego managed to calculate it, the car was of course already gone.

The problem statement mentions that Lego does not have to run perpendicular to the road. However, running toward the car is obviously not advantageous for him; therefore, let us denote the distance between the nearest point on the opposite side of the road and the point where he arrives as x. If he reaches this point sooner than the car, we can claim that he managed to cross before it. At the same time, we can assume that Lego will run at a slower speed than the car's speed. For this reason, if the car did not catch him at this point, it did not catch him earlier either. Consequently, there is no reason for Lego to follow a zig-zag path; the goal is simply to reach this specific point as soon as possible.

The distance of the car to the point where Lego leaves the road is $a + x$, so the car will be there in a time $(a + x)/v$. Lego's distance to this point is $\sqrt{x^2 + y^2}$, and therefore, he must run at a speed $v_\mathrm{L} = v\sqrt{x^2 + y^2}/(a + x)$.

It remains to find out for which x the required speed is minimal. To determine this speed, we differentiate v_L with respect to x

$$\frac{\mathrm{d}v_\mathrm{L}}{\mathrm{d}x} = v\frac{\frac{x(a+x)}{\sqrt{x^2+y^2}} - \sqrt{x^2 + y^2}}{(a + x)^2}\,,$$

and we look for where the derivative equals 0. This happens precisely when the numerator is zero

$$\frac{x\,(a + x)}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2} = 0\,,$$

$$xa + x^2 = x^2 + y^2\,,$$

$$x = \frac{y^2}{a}\,.$$

We substitute back into the expression for the required speed

$$v_\mathrm{L} = v\frac{\sqrt{x^2+y^2}}{a+x} = v\frac{\sqrt{\frac{y^4}{a^2}+y^2}}{a+\frac{y^2}{a}} = v\frac{y\sqrt{y^2+a^2}}{a^2+y^2} = v\frac{y}{\sqrt{y^2+a^2}}\,.$$

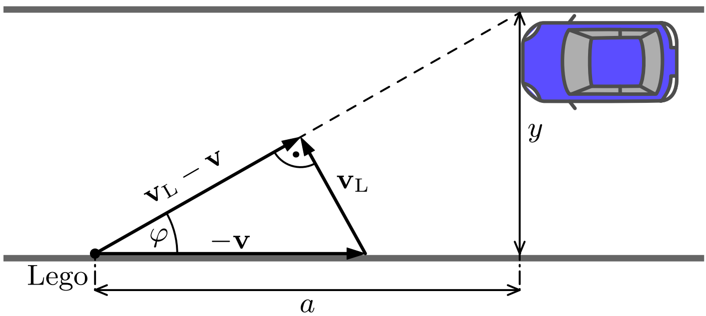
Solution in the car's reference frame

We can also solve this problem elegantly in the car's reference frame. In this frame, the car's position will not change over time. Let us introduce an x-axis parallel to the road such that the car is located at coordinate $x = 0$ and Lego is initially at coordinate $x = a$. Lego manages to run across the road if and only if he is located at a non-negative x-coordinate during his entire motion.

The limiting case, similar to the previous solution, will be motion such that Lego reaches the other side of the road at the same instant the car appears at the same location. In this limiting case, Lego will be located exactly at coordinate $x = 0$ on the other side of the road. Since it is not advantageous for Lego to zig-zag in the frame attached to him, his trajectory in the car's reference frame will certainly also be a straight line segment.

We can thus precisely draw the trajectory of Lego's motion in the frame attached to the car – it suffices to connect Lego's initial position and the car's position on the other side of the road with a line segment. Lego's velocity in the car's reference frame must have exactly this direction. Furthermore, we also know that Lego's velocity vector in this reference frame can be decomposed into the sum of the velocity vector $\mathbf{v}_\mathrm{L}$, with which he moved in the ground's reference frame, and the velocity vector $-\mathbf{v}$, where $\mathbf{v}$ is the car's velocity vector, also in the ground's reference frame.

We can perform this vector addition graphically, as in the figure below. We know that the resultant $-\mathbf{v} + \mathbf{v}_\mathrm{L}$ must define the line connecting Lego and the front of the car. We thus have a line and a point (the end of the vector $-\mathbf{v}$) which we want to connect with the shortest possible segment. And that is precisely the perpendicular to the given line from the end of the vector $-\mathbf{v}$.



In this way, we geometrically obtain a pair of right-angled triangles with one common angle φ. We can easily calculate the hypotenuse of the triangle along which Lego moves using the Pythagorean theorem as $\sqrt{y^2 + a^2}$. From there, we obtain an equation for the magnitudes of the individual velocities and the ratios of the side lengths in the given triangles

$$\sin\varphi = \frac{v_\mathrm{L}}{v} = \frac{y}{\sqrt{y^2 + a^2}}\,,$$

meaning the minimum speed with which Lego must move is

$$v_\mathrm{L} = v\frac{y}{\sqrt{y^2 + a^2}}\,.$$

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Problem EA ... kaon decay

A kaon particle with total energy $E_\mathrm{K} = 500\,\mathrm{MeV}$ *decays into two identical pions with the same energy. What will be the angle* α *between the directions in which the pions fly apart? The rest mass of the kaon is* $m_\mathrm{K} = 498\,\mathrm{MeV}/c^2$ *and the rest mass of the pion is* $m_\pi = 135\,\mathrm{MeV}/c^2$*.*
Hint: *You certainly know the famous relation* $E = mc^2$*. This can also be rewritten in the form* $E = \sqrt{m_0^2c^4 + p^2c^2}$ *where* m_0 *is the rest mass. Make use of this.*

Petr was practicing particle physics.

Let us denote the energy of a pion by E_π and the momenta of the pions by $\mathbf{p}_{\pi,1}$ and $\mathbf{p}_{\pi,2}$. First, we adjust the units. In particle physics, natural units are commonly used, in which we set $c = 1$ and write mass and momentum in units of energy. So, we redefine

$$m \equiv m_0c^2\,,$$
$$\mathbf{p} \equiv \mathbf{p}c\,,$$

specifically, for the rest masses of the kaon and the pion, we have

$$m_\mathrm{K} = 498\,\mathrm{MeV}\,,$$
$$m_\pi = 135\,\mathrm{MeV}\,.$$

Thanks to the relation

$$E = \sqrt{m^2 + p^2}$$

and the fact that the pions have the same energies, we know that the pions must have the same magnitude of momentum, which we denote uniformly by p_π. The law of conservation of momentum further gives

$$\mathbf{p}_\mathrm{K} = \mathbf{p}_{\pi,1} + \mathbf{p}_{\pi,2}\,,$$
$$p_\mathrm{K}^2 = 2p_\pi^2 + 2p_\pi^2\cos\alpha\,.$$

In the second equation, we computed the square of the magnitude of the momentum and used the formula $\mathbf{p}_{\pi,1}\cdot\mathbf{p}_{\pi,2} = p_\pi^2\cos\alpha$. From this, we express

$$p_\pi^2 = \frac{p_\mathrm{K}^2}{2\left(1 + \cos\alpha\right)}\,.$$

From the law of conservation of energy, it follows that

$$E_\mathrm{K}^2 = (2E_\pi)^2 \quad \Rightarrow \quad E_\mathrm{K}^2 = 4\left(m_\pi^2 + p_\pi^2\right) ,$$

from which, after substituting for p_π^2, we can express

$$\cos\alpha = \frac{E_\mathrm{K}^2 - 2m_K^2 + 4m_\pi^2}{E_\mathrm{K}^2 - 4m_\pi^2} .$$

By substitution and taking the inverse cosine, we obtain

$$\alpha \doteq 168^\circ .$$

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Problem EB ... three spheres

Consider three steel spheres, each with a mass of $m = 300\,\mathrm{g}$, attached to massless strings of length $L = 75\,\mathrm{cm}$. The other ends of the strings are connected at a single point. Each sphere carries a charge of $q = 5.0\,\mu\mathrm{C}$. When the entire system is suspended from the point where the strings are joined, what is the area of the horizontal triangle formed by the spheres? Feel free to solve the problem approximately or numerically.

Petr reminisced about electromagnetism course.

By symmetry, we can conclude that the triangle formed by the charges is equilateral. Let us denote the length of its side by a. Each charge then exerts a force on each of the others, given by

$$F_\mathrm{e} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2} = \frac{kq^2}{a^2}$$

which is directed away from the center of the triangle. The total force acting on any single charge is not simply twice the magnitude of the force exerted by one other charge—due to symmetry, the force components acting in opposite directions cancel out, leaving only the component along the axis of the triangle. The magnitude of this total force is given by

$$F_\mathrm{tot} = 2F_\mathrm{e}\cos 30^\circ = \frac{kq^2\sqrt{3}}{a^2} .$$

For the system to be in equilibrium, the length a must be such that the resulting force on a sphere is directed along the string tension. If this were not the case, a nonzero torque would act on the spheres, and the system would not be in equilibrium. The height of the triangle is

$$v = \frac{\sqrt{3}}{2}a ,$$

and the height of the resulting pyramid is

$$V = \sqrt{\frac{3L^2 - a^2}{3}} .$$

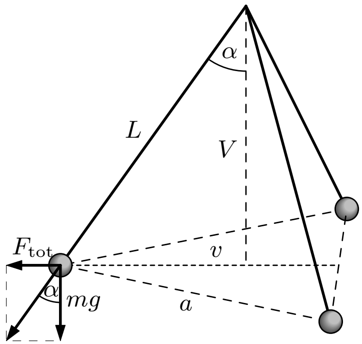

Figure 2: Diagram of the system in the equilibrium, including the forces acting on one of the spheres.

Here we have used a property of an equilateral triangle—the center of its circumscribed circle (which coincides with the centroid and the center of the inscribed circle) lies at 2/3 of the height measured from the vertex. The force condition mentioned above, therefore, gives

$$\tan\alpha = \frac{\frac{kq^2\sqrt{3}}{a^2}}{mg} = \frac{a}{\sqrt{3L^2 - a^2}}\,,$$

where α is the angle by which the strings deviate from the vertical axis. By rearranging the equation and introducing the notation $b \equiv a^2$, we obtain a cubic equation

$$b^3 + b\frac{3q^4}{16\pi^2 m^2 g^2 \varepsilon_0^2} - \frac{9q^4 L^2}{16\pi^2 m^2 g^2 \varepsilon_0^2} = 0\,,$$

$$\frac{m^2 g^2}{3k^2 q^4} b^3 + b - 3L^2 = 0\,.$$

This equation can be solved numerically using a calculator. Since it is an equation from which we can easily isolate b, an iterative method suggests itself. This method is based on rewriting the equation in the form $b = f(b)$ and then repeatedly substituting the most recently computed result b' back into the function f until the result no longer changes significantly. There are two possible approaches—either expressing b from the linear term or from the cubic term. If we were to express b from the linear term, we would quickly find that the resulting sequence does not converge. It is therefore advantageous to express b from the cubic term:

$$b = \sqrt[3]{\frac{C - b}{A}} \equiv f(b)\,,$$

where

$$A = \frac{m^2 g^2}{3k^2 q^4} \doteq 57.184\,\mathrm{m}^{-4}\,,$$

$$C = 3L^2 \doteq 1.687\,5\,\mathrm{m}^2.$$

On a calculator, the iteration can be carried out efficiently by first substituting an initial guess for b (for example, 0.1), evaluating the expression, and then replacing every occurrence of b on the right side of the equation with the calculator variable `ANS`.

$$\mathtt{ANS}_{n+1} = \sqrt[3]{\frac{C - \mathtt{ANS}_n}{A}},$$

where $\mathtt{ANS}_n$ is the result after the nth iteration (after the nth press of `=`). After several iterations, we obtain

$$b \doteq 0.2902\,\mathrm{m}^2\,.$$

The area of the triangle is then

$$S = \frac{\sqrt{3}}{4} b \doteq 0.13\,\mathrm{m}^2\,.$$

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Problem EC ... the shrunken Moon

In the movie Despicable Me, *the main character Gru shrinks the Moon and steals it. Imagine if instead of stealing it, Gru would have merely instantly shrunk the Moon such that the ratio of its new mass to its original mass was $m/M = 4/5$, while preserving the direction and magnitude of its total momentum and letting it continue orbiting the Earth. What would its new orbital period τ be? Assume that the Moon orbits the Earth along a circular trajectory with a radius $R = 3.844 \cdot 10^8$ m and that the mass of the Moon is significantly smaller than the mass of the Earth.* *Peter was watching the Minions*

First, we use the law of conservation of energy, which we can write for the shrunken Moon in the form

$$\frac{1}{2} m \|\mathbf{v}\|^2 - \frac{GmM_\oplus}{r} = E\,.$$

Since the momentum is conserved, we can determine the Moon's velocity immediately after its mass is changed:

$$mv = MR\frac{2\pi}{T} \quad \Rightarrow \quad v = \frac{2\pi}{T}\frac{M}{m}R\,.$$

We can express the period T using Kepler's third law. It states that for any body orbiting another (significantly heavier) body, the following holds:

$$\frac{a^3}{T^2} = \text{const.}\,,$$

where T is the orbital period and a is the semi-major axis of the ellipse along which the body orbits. We can calculate the unknown constant using the fact that the same relationship applies to a circular orbit, where the gravitational force cancels out the centrifugal force and $a = R$. Thus

$$m\left(\frac{4\pi^2}{T^2}\right)R = \frac{GmM_\oplus}{R^2} \quad \Rightarrow \quad \frac{R^3}{T^2} = \frac{GM_\oplus}{4\pi^2}\,,$$

from which we obtain

$$T = \frac{2\pi}{\sqrt{GM_\oplus}} R^{3/2} \quad \Rightarrow \quad v = \sqrt{\frac{GM_\oplus}{R}\left(\frac{M}{m}\right)}\,.$$

Thanks to this, we can express the constant[1] E/m as

$$\frac{E}{m} = \frac{1}{2}\frac{GM_\oplus}{R}\left(\left(\frac{M}{m}\right)^2 - 2\right).$$

In polar coordinates, we can express the square of the velocity magnitude in general as

$$\|\mathbf{v}\|^2 = \dot{r}^2 + r^2\dot{\varphi}^2\,,$$

where $\dot{r}^2$ and $r^2\dot{\varphi}^2$ are the squares of the instantaneous radial and tangent velocities. When the Moon is at perigee or apogee, the radial component of velocity is zero. At perigee or apogee, the law of conservation of energy gives us the equation

$$\frac{1}{2}mr^2\dot{\varphi}^2 - \frac{GmM_\oplus}{r} = E \quad \Rightarrow \quad r^3\dot{\varphi}^2 - \frac{2E}{m}r - 2GM_\oplus = 0\,.$$

Let us now express the instantaneous angular velocity $\dot{\varphi}$. Kepler's second law (or the law of conservation of angular momentum) tells us that

$$\frac{1}{2}r^2\dot{\varphi} = \frac{\pi R^2}{T}\frac{M}{m} = \text{const.}$$

and thus we can express

$$\dot{\varphi} = \frac{2\pi}{T}\frac{M}{m}\frac{R^2}{r^2} = \frac{M}{m}\frac{\sqrt{GM_\oplus R}}{r^2}\,.$$

Substituting this into the relation we obtained earlier from the law of conservation of energy at perigee or apogee, we get a quadratic equation for r

$$r^2 + \frac{2R}{\left(\frac{M}{m}\right)^2 - 2}r - \frac{\left(\frac{M}{m}\right)^2 R^2}{\left(\frac{M}{m}\right)^2 - 2} = 0\,,$$

whose solutions are

$$r = \begin{cases} \frac{(M/m)^2}{2-(M/m)^2}R = 1.373\cdot 10^9\,\mathrm{m} \\ R = 3.844\cdot 10^8\,\mathrm{m} \end{cases}$$

Note that one of the results corresponds to the original distance of the Moon from Earth R — meaning that immediately after the mass change, the Moon is at perigee. We could have noticed this earlier because, immediately after the mass change, the radial component of its velocity is zero. Therefore, we define r to be the distance at apogee; we can then determine the size of the semi-major axis of the ellipse along which the Moon newly orbits as

$$a = \frac{r+R}{2} = \frac{1}{2-\left(\frac{M}{m}\right)^2}R = \frac{16}{7}R\,.$$

[1] It is indeed a constant because energy is conserved.

We can now substitute this result into Kepler's third law

$$\frac{a^3}{\tau^2} = \frac{GM_\oplus}{4\pi^2}\,,$$

from which, by expressing τ, we get

$$\tau = \left(\frac{16}{7}\right)^{3/2} T = \left(\frac{16}{7}\right)^{3/2} \frac{2\pi}{\sqrt{GM_\oplus}} R^{3/2} \doteq 94.9\,\mathrm{d}\,.$$

Solution using the vis-viva equation

Before shrinking, the orbital speed of the Moon is

$$v_0 = \frac{2\pi R}{T} \quad \Rightarrow \quad v_0^2 = \frac{4\pi^2 R^2}{T^2} = \frac{GM_\oplus}{R}\,,$$

where we used Kepler's third law among algebraic manipulations.

$$\frac{R^3}{T^2} = \frac{GM_\oplus}{4\pi^2} = \mathrm{const.}\,.$$

The velocity v_0 is also referred to as the *first cosmic velocity*.

According to the problem statement, momentum is conserved during the shrinking; therefore, for the mass m and the velocity after shrinking v_1, the following holds:

$$v_1 m = v_0 M\,.$$

Then, by rearranging, we obtain:

$$v_1 = \frac{M}{m} v_0 \quad \Rightarrow \quad v_1^2 = \left(\frac{M}{m}\right)^2 v_0^2 = \left(\frac{M}{m}\right)^2 \frac{GM_\oplus}{R}\,.$$

For motion along elliptical and hyperbolic trajectories, the *vis-viva* equation applies:

$$v^2 = G\mathcal{M}\left(\frac{2}{r} - \frac{1}{a}\right)\,,$$

which gives the relationship between the general distance r from the central body of mass $\mathcal{M} \gg m$ on an orbit with semi-major axis a (for hyperbolic trajectories, this axis has a negative sign) and the magnitude of the velocity v at a given moment. In our case, the Moon orbits the Earth with mass $M_\oplus$; at the moment of shrinking, it has velocity v_1 and distance R from the Earth, so the following holds:

$$GM_\oplus\left(\frac{2}{R} - \frac{1}{a}\right) = v_1^2 = \left(\frac{M}{m}\right)^2 \frac{GM_\oplus}{R}$$

and after algebraic manipulations

$$a = \frac{R}{2 - \left(\frac{M}{m}\right)^2} = \frac{16}{7} R\,.$$

This result is consistent with the previous method.

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Problem ED ... anti-reflective coating

To prevent light from reflecting off glasses (which, for example, looks bad in photographs), we can apply an anti-reflective coating. Suppose we wish to apply an anti-reflective coating made of a material with a refractive index $n = 1.38$ onto glasses with a refractive index $N > n$. What is the minimum thickness d of the layer required so that no light is reflected at normal incidence, considering only a single reflection? Assume a standard wavelength of $\lambda = 550\,\mathrm{nm}$.

Petr wants to look good in photographs.

Since $N > n$ and $n > n_0$, the phase changes by π during both reflections — on the lens glass and on the anti-reflective coating. When the reflected rays subsequently interfere, the phase difference due to reflection is $\pi - \pi = 0$, and thus plays no role. We want the wave traversing the anti-reflective layer to acquire a phase delay of $(2p+1)\pi$, where $p \in \mathbb{N}_0$ — this way, the wave that passes through the layer and reflects back will be exactly in anti-phase with the wave reflected from the coating's surface, resulting in destructive interference. As the ray passes through the anti-reflective layer, it travels a distance of $2d$; the optical path length is therefore $2dn$. Thus, we require

$$2dnk = (2p+1)\,\pi\,,$$

where k is the angular wavenumber in vacuum. We can express this using the wavelength λ as

$$k = \frac{2\pi}{\lambda} \quad \Rightarrow \quad 2dn\frac{2}{\lambda} = (2p+1)\;.$$

We obtain the thinnest layer if the phase difference is minimal, that is, $p = 0$. By substituting for k and p, and expressing d, we then have

$$d = \frac{\lambda}{4n} \doteq 99.6\,\mathrm{nm}\,.$$

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Problem EE ... physical pendulums with stiff coupling

Consider two rods of length $l = 15\,\mathrm{cm}$, each suspended by one of its ends and able to rotate freely about the suspension point. These suspension points are at the same height, their mutual distance is equal to l, and the free ends of the rods are connected by another rod also of length l. All three rods have mass $m = 300\,\mathrm{g}$. What is the period of small oscillations if we set the system into motion in the plane in which the rods lie? The system is located in a gravitational field with acceleration g.

Lego's problem was physics-ified.

Since the system consists of three independently moving parts executing different harmonic motions, we will not use the standard procedure based on an equation of motion, but instead focus on energies. We therefore express the potential and kinetic energy as functions of the angular displacement of the vertical rods φ and their angular velocity $\dot{\varphi}$.

When the hanging rods are displaced by an angle φ from the vertical direction, the centers of mass of the two hanging rods are raised by $(l/2)(1 - \cos\varphi)$ from the equilibrium position, and the connecting rod is raised by $l(1 - \cos\varphi)$. The total potential energy is

$$E_\mathrm{p} = mgl\left(2\frac{1}{2} + 1\right)(1 - \cos\varphi) \approx 2mgl\frac{\varphi^2}{2}\,,$$

where we have used the second order Taylor expansion for the cosine as $\cos\varphi \approx 1 - \varphi^2/2$.

The hanging rods rotate about their suspension points. The moment of inertia of each of these rods with respect to the suspension point is $ml^2/3$, so when they move with angular velocity $\dot{\varphi}$, the kinetic energy of each of them is

$$\frac{1}{6}ml^2\dot{\varphi}^2 \,.$$

The connecting rod does not rotate; it remains horizontal at all times, so it suffices to use the formula for translational kinetic energy with the velocity of its center of mass. The center of mass moves along a circle of radius l with angular velocity $\dot{\varphi}$, that is, with speed $l\dot{\varphi}$. The kinetic energy is then $(1/2)ml^2\dot{\varphi}^2$. Altogether, the kinetic energy is

$$E_\mathrm{k} = \left(\frac{1}{3} + \frac{1}{2}\right) ml^2\dot{\varphi}^2 = \frac{5}{6}ml^2\dot{\varphi}^2 \,.$$

Now, by analogy with the linear harmonic oscillator, we rewrite the energies in the form

$$E_\mathrm{p} = \frac{1}{2}k_\mathrm{ef}q^2 \,,$$
$$E_\mathrm{k} = \frac{1}{2}m_\mathrm{ef}\dot{q}^2 \,,$$

which yields that the effective stiffness of our pendulum is $k_\mathrm{ef} = 2mgl$ and the effective mass is $m_\mathrm{ef} = 5ml^2/3$. Since we use an angle as the coordinate, these two quantities have the dimensions of torque and moment of inertia, respectively. In any case, it only remains to substitute into the formula for the period of small oscillations

$$T = 2\pi\sqrt{\frac{m_\mathrm{ef}}{k_\mathrm{ef}}} = 2\pi\sqrt{\frac{\frac{5}{3}ml^2}{2mgl}} = 2\pi\sqrt{\frac{5l}{6g}} \doteq 0.71\,\mathrm{s} \,.$$

Finally, note that the problem could also be solved using the standard procedure for a physical pendulum, because the motion of the individual components is independent of their position. We can therefore virtually shift them so that their axes of rotation coincide at a single point. Such a resulting pendulum would have mass $M = 3m$ and total moment of inertia $I = ml^2/3 + ml^2/3 + ml^2 = 5ml^2/3$ (since the horizontal rod does not spin, it has the moment of inertia of a point mass). The distance of the center of mass of this pendulum from the common axis would be calculated as the average of the distances of the centers of mass of the individual components, $L = (l/2 + l/2 + l)/3 = 2l/3$. These values can again be substituted into the tabulated formula, yielding the same result as with the previous method

$$T = 2\pi\sqrt{\frac{I}{MgL}} = 2\pi\sqrt{\frac{5l}{6g}} \,.$$

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Problem EF ... inside of a glowing sphere

Imagine you are inside a hollow sphere with a radius equal to the Sun's radius $R_\odot$, positioned at a distance $R_\odot/2$ from its center. The walls of the sphere have the surface temperature of the Sun $T = 5\,800\,\mathrm{K}$. Your body can be approximated as a sphere with a cross-sectional area $S = 0.70\,\mathrm{m}^2$ and a mass $m = 70\,\mathrm{kg}$, and assume that you absorb $\eta = 55\,\%$ of the incident radiation. Assume that each surface element of the sphere emits radiation isotropically. Determine the magnitude and direction of the force exerted on you by the radiation pressure emitted by the walls of the sphere. *Vlado got off-topic at the Christmas gathering.*

Let us consider an element of a sphere that emits radiation isotropically. The radiation intensity I of this element decreases according to the inverse-square law, which states that I decreases with the second power of the distance from the source

$$I \propto r^{-2}\,.$$

The force with which radiation acts on a body is caused by the change in the momentum of photons during their interaction with the body. In the case of complete absorption of photons, this is a perfectly inelastic collision; in the case of complete reflection, it is a perfectly elastic collision. The radiation pressure $\mathcal{P}$ (the radiation force per unit area) is therefore proportional to the change in photon momentum p, and thus[2]

$$\mathcal{P} \propto \Delta p \propto p \propto E_{\mathrm{photon}} \propto I \propto r^{-2}\,.$$

The radiation pressure emitted by an element of the surface of a sphere is proportional to r^{-2}. For an exact derivation of the force with which radiation acts on a body, we would have to take the geometry of the body into account and separately compute the contributions of absorbed and reflected light; however, all these effects are ultimately proportional to $\mathcal{P}$, so $F \propto \mathcal{P} \propto r^{-2}$ holds. This problem is therefore mathematically equivalent to finding the force exerted by a charged sphere on a body inside it that carries a charge of the same sign. According to Gauss's law, this force is zero, hence

$$F = 0\,\mathrm{N}\,.$$

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Problem EG ... got to be the best pirate I have ever seen

Jack Sparrow is sailing into a harbor with velocity v_0 on a leaking boat. Water is flowing into the boat at a constant rate Q, the boat's total volume is V, and its mass including the pirate is m_0. Jack is currently counteracting the intake using a bucket, but to preserve his pirate aura, he intends to stop at a certain moment and let the boat drift to the pier, where it will sink. At what distance from the pier should he stop using the bucket? Neglect resistive forces.
"So it would seem" said Petr.

Due to the water inflow, the boat's mass at time t after Jack stops removing water is given by

$$m(t) = m_0 + Q\rho t\,.$$

[2] A complete derivation of this relation is given in the problem *repulsive light.*

Archimedes' principle states that the buoyant force acting on the boat is proportional to the submerged volume. It is maximized when the entire boat is submerged, that is,

$$F_{\max} = V\rho g\,.$$

To prevent the boat from sinking, the maximum buoyant force must always exceed the force of gravity; therefore, the following must hold:

$$V\rho g - (m_0 + Q\rho t)\,g \geq 0\,.$$

In the limiting case where the forces balance out exactly at the limiting time T, we obtain the condition

$$T = \frac{V\rho - m_0}{\rho Q}\,.$$

During the time T from the moment Jack Sparrow stops removing the incoming water, the boat must cover a distance d and reach the pier. However, since the boat's mass changes, its velocity changes as well. The law of conservation of momentum yields

$$m_0 v_0 = m(t)\,v(t)\ ,$$

from which, using $m(t)$, we can express the boat's velocity $v(t)$ as a function of time:

$$v(t) = \frac{m_0 v_0}{m_0 + Q\rho t}\,.$$

We now simply integrate this from the initial time 0 to time T. We have

$$d = \int_0^T \frac{m_0 v_0}{m_0 + Q\rho t}\,\mathrm{d}t\,,$$

$$u = m_0 + Q\rho t \quad \Rightarrow \quad d = \frac{m_0 v_0}{Q\rho} \int_{m_0}^{m_0 + Q\rho T} \frac{1}{u}\,\mathrm{d}u\,,$$

$$d = \frac{m_0 v_0}{Q\rho}\left[\ln u\ \right]_{m_0}^{m_0 + Q\rho T}\ ,$$

which, after substituting for T, yields

$$d = \frac{m_0 v_0}{Q\rho} \ln\left(\frac{V\rho}{m_0}\right)\,.$$

A common error in solving this problem is assuming that energy is conserved, which does not hold here, that is,

$$\frac{1}{2} m_0 v_0^2 \neq \frac{1}{2} m v^2\,.$$

This is because the water effectively undergoes a perfectly inelastic collision with the boat.

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Problem EH ... grilled Roman, Sicilian-style

Archimedes is said to have constructed a machine made of polished copper mirrors to defend Syracuse, intended to ignite enemy ships.

Imagine such a machine, suppose it consists of a copper plate bent into a parabolic shape given by the equation $\eta = \pi\xi^2$, where π is a parameter. We can tune the value of the parameter π, and thus the focusing of the "mirror", by turning a hand crank. The rotation of the crank is linearly related to the parameter π by $\pi = \alpha\vartheta + \vartheta_0$, where ϑ is the rotation of the crank in radians and $\alpha = 2.0 \cdot 10^{-5}\,\mathrm{m^{-1}{\cdot}rad^{-1}}$. The ship must be targeted, i.e., placed at the focus of the parabola. The ship approaches us with uniform linear motion at a speed of $\nu = 15\,\mathrm{km{\cdot}h^{-1}}$, and at time $\tau_0 = 0$, it is targeted at a distance of $\varphi_0 = 1.0\,\mathrm{km}$. At what speed must we turn the crank at time $\tau = 3.0\,\mathrm{min}$, so that the ship remains targeted?

Petr was watching a video about the Punic wars.

In the first phase, we should determine how the focal distance φ relates to the single parameter of the parabola, π. If we do not know this relationship, it does not matter; we will derive it. A parabola is defined as a curve whose points are all equidistant from the focus Φ and the directrix. Let us define a coordinate system $[\xi, \eta]$ with the origin at the vertex of our mirror and the η axis pointing toward the ship. In our situation, Φ is located at the point $\Phi = [0, \varphi]$. The directrix has the equation $\eta = -\varphi$, which we easily determine from the fact that the point $[0, 0]$ must be at a distance φ from it and must lie below the parabola. Let there be a point $A = [\xi, \eta]$ on the parabola and a point B on the directrix directly below it. The distances $|\Phi A|$ and $|AB|$ are given by

$$|\Phi A| = \sqrt{\xi^2 + (\eta - \varphi)^2}\,,$$
$$|AB| = (\eta + \varphi)\ .$$

By the definition of a parabola, the equality

$$|\Phi A| = |AB|$$

must hold, which, using the relations above, substituting for η, and expressing φ, yields

$$\varphi = \frac{1}{4\pi}\,.$$

The distance to the required focal point varies with time as

$$\varphi = \varphi_0 - \nu\tau\,,$$

Substituting for π in the expression for φ above, we find that

$$\varphi = \frac{1}{4\left(\alpha\vartheta + \vartheta_0\right)}\,.$$

We thus have the equality

$$\frac{1}{4\left(\alpha\vartheta + \vartheta_0\right)} = \varphi_0 - \nu\tau\,.$$

Expressing the rotation angle of the crank ϑ, we obtain the time dependence

$$\vartheta(\tau) = \frac{1}{4\alpha\left(\varphi_0 - \nu\tau\right)} - \frac{\vartheta_0}{\alpha}\,.$$

To determine the required angular velocity of the crank rotation ω, we differentiate this equation with respect to time, obtaining

$$\omega = \dot{\vartheta} = \frac{\nu}{4\alpha}\frac{1}{(\varphi_0 - \nu\tau)^2}$$

and by evaluating the expression for the given values, we obtain

$$\omega \doteq 0.83\,\mathrm{rad{\cdot}s^{-1}}\,.$$

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Problem FA ... Starcraper

Marek wants to build a skyscraper at the latitude $\varphi = 15^\circ$ that reaches the stars. How tall should it be to slow down the Earth's rotation by $1\,\%$? Consider the Earth to be a homogeneous sphere of mass M, which remains unchanged by this construction, since Marek extracts the mass from a white hole. Assume that the mass of the building is initially at rest, and it is precisely its rotation that slows down the Earth. Further assume that the skyscraper is sufficiently thin, homogeneous, and has mass $m = 33 \cdot 10^{-4} M$.

Marek likes to look at things from a different perspective.

During the construction of the skyscraper, the Earth's angular momentum L is conserved. Before the construction begins, this angular momentum is

$$L = J\omega = \frac{2}{5}MR^2\omega\,,$$

where J is the moment of inertia of the Earth before the construction, ω is its angular frequency, M is the mass of the Earth, and R its radius. We used the formula for the moment of inertia of a homogeneous sphere.

What is the moment of inertia of the system after the construction? It consists of two parts—the Earth, which has the same moment of inertia as before, plus the contribution of the skyscraper. Let us compute the moment of inertia of a homogeneous rod with linear mass density λ and length l about its center, inclined at an angle $\alpha = 90^\circ - \varphi$ from the axis of rotation, which is precisely our skyscraper.

We can imagine that the rod is composed of small elements of length $\mathrm{d}l$, each with mass $\mathrm{d}m = \lambda\,\mathrm{d}l$. If we consider a coordinate x in the direction of the rod, but perpendicular to the axis of rotation, the projection of the element along this coordinate is $\mathrm{d}x = \mathrm{d}l\cos\varphi$. From the definition, the moment of inertia of the rod is

$$J'_\mathrm{v} = \int x^2\,\mathrm{d}m = \int_{-l\cos\varphi/2}^{l\cos\varphi/2} x^2 \frac{\lambda}{\cos\varphi}\,\mathrm{d}x = \frac{1}{12}\,(\lambda l)\,l^2\cos^2\varphi\,,$$

and since $m = \lambda l$ is the mass of the rod, for a rod perpendicular to the axis of rotation we would obtain the well-known formula $(1/12)ml^2$, confirming our result.

However, the center of mass of our rod is at a distance $d = (R + l/2)\cos\varphi$ from the Earth's axis of rotation, so by the parallel axis theorem, its moment of inertia for rotation about this axis is $J_\mathrm{v} = J'_v + md^2$.

Let ω' be the angular frequency after construction. Then the equation for conservation of the total angular momentum is

$$\begin{aligned}
L = J\omega &= (J + J_\mathrm{v})\omega' \,, \\
J\omega &= \left(J + \frac{1}{12} m l^2 \cos^2\varphi + m\left(\left(R + \frac{l}{2}\right)^2 \cos^2\varphi\right)\right)\omega' \,, \\
0 &= \frac{1}{3} l^2 + Rl + \left(R^2 - \frac{J}{m\cos^2(\varphi)}\left(\frac{\omega}{\omega'} - 1\right)\right) \,, \\
l &= \frac{3R}{2}\left(-1 + \sqrt{1 - \frac{4}{3}\left(1 - \frac{J}{R^2}\frac{1}{m\cos^2(\varphi)}\left(\frac{\omega}{\omega'} - 1\right)\right)}\right) = \\
&= \frac{R}{2}\left(-3 + \sqrt{9 - 12 + \frac{24M}{495 m\cos^2\varphi}}\right) = \frac{R}{2}\left(-3 + \sqrt{\frac{8M}{165 m\cos^2\varphi} - 3}\right) \,,
\end{aligned}$$

where we chose the physically relevant positive root. In the last step we substituted for J and used $\omega'/\omega = 99/100$ from the problem statement. Substituting the values for m and φ yields $l \doteq 1\,800\,\mathrm{km}$.

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Problem FB ... oscillating massive pulley

Consider a homogeneous pulley of mass m in the shape of a disk with radius r. We suspend this pulley from the ceiling such that on one side, the rope is attached to the ceiling directly, and on the other side, it is attached via a spring of stiffness k. We pull the pulley slightly downwards from its equilibrium position. What is the period of small oscillations? Assume that the pulley does not slip on the rope.

Presumably, since there was a lack of problems involving oscillations, Lego proposed this one.

In the equilibrium position, the rope is under a tension of $mg/2$, which is also the force exerted by the spring on the rope.

When we pull the pulley downwards such that its center moves down by x, the spring must extend by $\Delta y = 2x$ relative to its equilibrium length (since the rope does not extend at all on the other side). Consequently, the force it exerts on the pulley increases by $\Delta F_\mathrm{k} = k\Delta y = 2kx$ compared to the equilibrium case, meaning the total spring force becomes $F_\mathrm{k} = mg/2 + 2kx$. This is precisely the force pulling one side of the pulley upwards. Since the pulley has mass, we do not know the force pulling the other side upwards; let us denote this force as $T = mg/2 + \Delta T$ (we could denote this force simply as T and use that in calculations, but this decomposition is more practical).

The total force acting on the pulley is therefore

$$F = F_\mathrm{k} + T - F_g = mg/2 + 2kx + mg/2 + \Delta T - mg = 2kx + \Delta T \,,$$

where we consider the upward direction to be positive. At the same time, it is hopefully clear why the substitution $T = mg/2 + \Delta T$ is so useful. Thus, the acceleration of the pulley is $a = (2kx + \Delta T)/m$.

Let us now look at the torque. The weight of the pulley has zero torque with respect to its center. The ropes on both sides have a lever arm of r, but they rotate the pulley in opposite directions. We choose the positive direction of rotation as the one where the side of the pulley under the spring rotates upwards (since this is exactly what happens when the pulley moves in this direction). The torque is therefore

$$M = rF_{\mathrm{k}} - rT + 0F_g = r(mg/2 + 2kx) - r(mg/2 + \Delta T) = r(2kx - \Delta T)\,.$$

The moment of inertia of the disk is $I = mr^2/2$; thus, the angular acceleration is

$$\varepsilon = \frac{M}{I} = \frac{r(2kx - \Delta T)}{mr^2/2} = 2\frac{2kx - \Delta T}{mr}\,.$$

Since the rope on the side without the spring does not move, it is evident that the velocity of ascent and the tangential velocity must have the same magnitude (because the pulley essentially rolls up the rope). Therefore, the same relationship must hold between the acceleration and the "tangential acceleration"

$$a = \varepsilon r\,,$$

$$\frac{2kx + \Delta T}{m} = 2\frac{2kx - \Delta T}{m}\,,$$

$$2kx + \Delta T = 4kx - 2\Delta T\,,$$

$$\Delta T = \frac{2}{3}kx\,.$$

We can substitute for ΔT to express the acceleration

$$-\ddot{x} = a = \frac{2kx + \Delta T}{m} = \frac{8}{3}\frac{k}{m}x\,,$$

which is the equation of a linear harmonic oscillator with $\omega^2 = 8k/(3m)$, so the period of small oscillations is

$$T = 2\pi\sqrt{\frac{3}{8}\frac{m}{k}}\,.$$

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Problem FC ... motocross derby

Three motorcyclists Pepa, Vojta, and Marek, along with their motorcycles, are positioned at the vertices of an equilateral triangle with side length a. At time $t = 0$, they all simultaneously begin to pursue one another with speed v_0, with Pepa chasing Vojta, Vojta chasing Marek, and Marek chasing Pepa. However, they do not want to collide at full speed; thus, the closer they get to each other, the more they slow down. Therefore, their speed is directly proportional to their mutual distance $v(l) = (l/a)v_0$. How long does it take for the distance between Vojta and Marek to decrease to x (assuming $x < a$)? *Kubo already has a third motorcycle at home.*

First, it is necessary to realize that the motions of all three motorcyclists are mutually symmetric with respect to the center of the original triangle. Their mutual positions therefore always

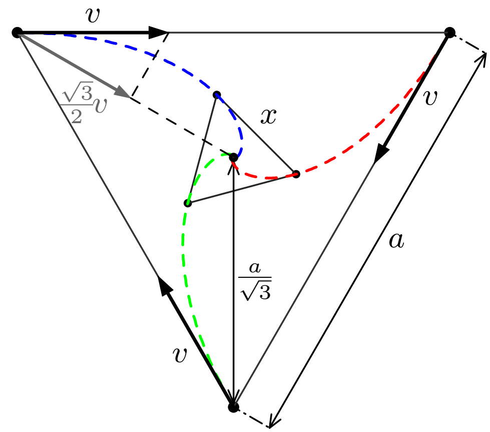

Figure 3: Diagram of the situation with the trajectories marked.

form the vertices of an equilateral triangle whose center remains fixed at the original location; however, the side length gradually decreases and the orientation changes.

Let us consider the radial distance of one of the motorcyclists from the center of the triangle. At time $t = 0$, it has the value $r_0 = a/\sqrt{3}$. Subsequently, the motorcyclist approaches the center in this direction with speed $v\sqrt{3}/2$ (the projection of his velocity onto the radial direction). Therefore,

$$\dot{r} = -\frac{\sqrt{3}}{2}\frac{l}{a}v_0 = -\frac{3}{2}\frac{r}{a}v_0 \,.$$

The decrease in $r(t)$ is thus directly proportional to its instantaneous value, which implies an exponential decrease

$$r(t) = r_0 \exp\left(-\frac{3}{2}\frac{v_0}{a}t\right) \,.$$

The requirement that the mutual distance of two motorcyclists becomes equal to x can be reformulated as the equality of the ratios r/r_0 and x/a. It then suffices to rearrange the resulting equation and solve for the desired time t,

$$\frac{r(t)}{r_0} = \exp\left(-\frac{3}{2}\frac{v_0}{a}t\right) \stackrel{!}{=} \frac{x}{a} \,,$$

$$t = \frac{2a}{3v_0}\ln\left(\frac{a}{x}\right) \,.$$

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Problem FD ... conical cup

Let us consider a cup in the shape of a hollow cone without a base, of height h, whose opening angle is α. We fill the cup with a liquid of density ρ all the way up to the rim. Because such a cup would stand poorly with its apex down, we quickly turn it upside down and place it on a table so that not a single drop of liquid spills out. What buoyant force acts on the cup? The liquid at the apex inside the cone is at atmospheric pressure.

Petr found this story in a Vietnamese textbook.

The force with which the liquid acts on the cup is caused by the hydrostatic pressure

$$p = z\rho g\,,$$

where z is the depth at which an element of the cup is located. The hydrostatic pressure is an "addition" to the atmospheric pressure, which acts on the cup from the outside and by which the liquid then acts back. The liquid at the very tip of the cup therefore acts on the cup only by atmospheric pressure. Since this constant contribution cancels out with the external pressure, we do not need to take it into account.

Consider a small element of the inner surface of the cup $\mathrm{d}S$; the force exerted on it by the liquid is $\mathrm{d}F = p\,\mathrm{d}S$. However, from symmetry, we see that when all these infinitesimal forces are summed up, the radial components cancel each other out and only the force in the upward direction remains. It therefore suffices to consider only the projection of $\mathrm{d}F$ in the upward direction; from simple geometry, we have

$$\mathrm{d}F_{\uparrow} = \mathrm{d}F \sin\frac{\alpha}{2}\,.$$

To obtain the net force, we must integrate

$$F = \int_S \mathrm{d}F_{\uparrow}\,.$$

We can describe the cup by two cylindrical parameters $z \in \langle 0, h\rangle$ and $\varphi \in \langle 0, 2\pi)$; the radius of the cup is then a dependent variable, for which we can derive

$$R = z\tan\frac{\alpha}{2}\,.$$

The final step before integration is to determine the form of the surface element $\mathrm{d}S$ expressed in the chosen coordinates.[3] If we consider an infinitesimally small rectangle cut out of the lateral surface of the cone with dimensions $\mathrm{d}x$ and $\mathrm{d}y$, where $\mathrm{d}x$ lies in the horizontal plane and $\mathrm{d}y$ is directed toward the apex, we have

$$\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y\,.$$

We now express $\mathrm{d}x$ and $\mathrm{d}y$ in terms of $\mathrm{d}\varphi$ and $\mathrm{d}z$. We have

$$\mathrm{d}x = R(z)\,\mathrm{d}\varphi = z\tan\frac{\alpha}{2}\,\mathrm{d}\varphi\,,$$

[3] Beside the stated procedure, this can be derived in a tedious mathematical way by computing the norm of the vector product between the derivatives of the vector that parametrizes the cone, as taught by the theory of surface integrals of the first kind.

and if we additionally notice that $\mathrm{d}z$ is in fact the projection of $\mathrm{d}y$ onto the vertical axis, we obtain

$$\mathrm{d}y = \frac{1}{\cos\frac{\alpha}{2}}\,\mathrm{d}z\,.$$

Substituting the previous results into the integral for F and simplifying, we obtain

$$F = \int_S p(z)\sin\frac{\alpha}{2}\,\mathrm{d}x\,\mathrm{d}y = \tan^2\frac{\alpha}{2}\rho g\int_0^h\int_0^{2\pi} z^2\,\mathrm{d}\varphi\,\mathrm{d}z\,.$$

Finally, we arrive at

$$F = \frac{2}{3}\pi\rho g\tan^2\left(\frac{\alpha}{2}\right)h^3\,.$$

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Problem FE ... fast-spinning kettle

Marek has a thin thermally insulating spherical shell of radius R and mass M filled with water. He spins up the sphere while keeping the water inside at rest. The sphere begins to slow down due to interaction with the water, and after a long time, Marek finds that the temperature inside the sphere has risen by ΔT. To what angular velocity did he spin up the sphere?

Assume that the water has a constant density ρ and specific heat capacity c_v. Due to friction, which also heats the water, angular momentum is not conserved.

Marek was wandering around Matfyz.

The rotational energy of the shell is converted into thermal energy, which heats the water. The rotational energy is

$$E_\mathrm{rot} = \frac{1}{2}J\omega^2\,,$$

where J is the moment of inertia of the spherical shell. The thermal energy is given by

$$E_\mathrm{therm} = mc_\mathrm{v}\Delta T = \frac{4}{3}\pi R^3\rho c_\mathrm{v}\Delta T\,,$$

where m is the mass of the water inside.

It remains to calculate the moment of inertia of the spherical shell. Since the shell is homogeneous, we define the surface density

$$\sigma = \frac{M}{S} = \frac{M}{4\pi R^2}\,,$$

where S is the surface area of the shell. Let us imagine slicing the sphere “horizontally” (perpendicular to the axis of rotation) into thin rings, whose height z ranges from $-R$ to R. Each of them has a radius r, determining the distance from the axis of rotation, and a height $\mathrm{d}z$. From the Pythagorean theorem, we have $r = \sqrt{R^2 - z^2}$ and the surface area of the ring is $\mathrm{d}S = 2\pi R\,\mathrm{d}z$. The mass of the ring is

$$\mathrm{d}m = \sigma\,\mathrm{d}S = \frac{M\,\mathrm{d}z}{2R}\,,$$

The moment of inertia of a single ring is

$$\mathrm{d}J = r^2\,\mathrm{d}m = \left(R^2 - z^2\right)\frac{M}{2R}\,\mathrm{d}z\,;$$

to calculate the total moment, we must sum up—or more precisely integrate—over all z, i.e.,

$$J = \frac{M}{2R}\int_{-R}^{R}\left(R^2 - z^2\right)\mathrm{d}z = \frac{M}{2R}\left(2R^3 - \frac{2}{3}R^3\right) = \frac{2}{3}MR^2\,.$$

It remains to substitute into the formula for energy, and we obtain

$$E_{\mathrm{rot}} = E_{\mathrm{therm}}\,,$$
$$\frac{1}{3}MR^2\omega^2 = \frac{4}{3}\pi R^3\rho c_{\mathrm{v}}\Delta T\,,$$
$$\omega = 2\sqrt{\frac{\pi R\rho c_{\mathrm{v}}\Delta T}{M}}\,.$$

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Problem FF ... definitional

Consider a homogeneous rod standing vertically upward on the Earth's surface. What must be the length of this rod so that its center of gravity, i.e., the point where the gravitational force effectively acts, is at a distance of 1.0 m *from its center of mass?*
Hint: *You may find it useful that for small x the approximation* $\ln\left(1+x\right) \approx x - x^2/2 + x^3/3$ *holds.* *Marek was contemplating his height.*

The center of mass of a homogeneous rod is at its midpoint, i.e., at a height $y_s = L/2$ above the ground.

The center of gravity is the average of the positions of the rod segments, weighted by the force acting on each segment. Since the rod is homogeneous, it has a constant linear density (mass per unit length) $\lambda = M/L$, where M is the mass of the rod and L is its length. The gravitational force acting on a segment of mass $\mathrm{d}m$ is

$$\mathrm{d}F_{\mathrm{g}} = G\frac{M_{\mathrm{Z}}\,\mathrm{d}m}{r^2} = G\frac{M_{\mathrm{Z}}\lambda\,\mathrm{d}r}{r^2}\,,$$

where r is the distance of the given segment from the center of the Earth, M_{Z} is the mass of the Earth, and $\mathrm{d}r$ is the length of the given segment.

The weighted average over the segments is then

$$r_{\mathrm{p}} = \frac{\int r\,\mathrm{d}F_{\mathrm{g}}}{\int \mathrm{d}F_{\mathrm{g}}} = \frac{\int_R^{R+L} r\frac{1}{r^2}\,\mathrm{d}r}{\int_R^{R+L}\frac{1}{r^2}\,\mathrm{d}r}\,,$$

where R is the radius of the Earth. Evaluating the integrals yields

$$r_{\mathrm{p}} = \frac{\ln\frac{R+L}{R}}{\frac{1}{R} - \frac{1}{R+L}} = \frac{R(R+L)}{L}\ln\left(1 + \frac{L}{R}\right).$$

To arrive at the result, we use the Taylor expansion of the logarithm, since $L \ll R$ will certainly hold. Then

$$\ln\left(1+\frac{L}{R}\right) \approx \frac{L}{R} - \frac{L^2}{2R^2} + \frac{L^3}{3R^3}\,.$$

Substituting, we obtain

$$\begin{aligned} r_\mathrm{p} &\approx \frac{R(R+L)}{L}\left(\frac{L}{R} - \frac{L^2}{2R^2} + \frac{L^3}{3R^3}\right) = \\ &= R\left(1+\frac{L}{R}\right)\left(1 - \frac{L}{2R} + \frac{L^2}{3R^2}\right) \approx \\ &\approx R + \frac{L}{2} - \frac{L^2}{6R}\,, \end{aligned}$$

where in the last step we neglected terms of the highest order in L/R. Let us look at the result—the height of the center of gravity is almost at a height $R + L/2$ above the center of the Earth, just like the center of mass; only the last term makes a difference, and this difference is supposed to be one meter.

By substitution, we obtain $L \approx 6.2\,\mathrm{km}$.

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Problem FG ... a new lamp

Vlado bought a new lamp, but as is often the case these days, new devices come only with a cable and no adapter. Vlado has a $U = 24\,\mathrm{V}$ adapter available, but the nominal voltage of the new lamp is $U_\mathrm{L} = 12\,\mathrm{V}$. He therefore decides to connect it to this adapter through a potentiometer, which he connects as a voltage divider. Assume that the lamp has power input $P_\mathrm{L} = 12\,\mathrm{W}$. Vlado cares about the planet and thus wants the efficiency of the entire system to be at least $\eta = 40\,\%$. Calculate the total resistance of the potentiometer at which the maximum current flows through the circuit. *Vlado was(n't) illuminated.*

Let us denote the total resistance of the potentiometer as R, the resistance of the portion of the potentiometer in the circuit branch as R_X (Fig. 4), and the resistance of the lamp as $R_\mathrm{L} = U_\mathrm{L}^2/P_\mathrm{L} = 12\,\Omega$. Since the voltage across both branches is U_L, according to Kirchhoff's first law, we have

$$I = \frac{U_\mathrm{L}}{R_\mathrm{L}} + \frac{U_\mathrm{L}}{R_\mathrm{X}}\,. \tag{3}$$

The total efficiency of the circuit is equal to the ratio of the lamp power P_L to the source power P, i.e.,

$$\eta = \frac{P_\mathrm{L}}{P} = \frac{\frac{U_\mathrm{L}^2}{R_\mathrm{L}}}{UI} \quad\Rightarrow\quad I = \frac{U_\mathrm{L}^2}{\eta U R_\mathrm{L}}\,.$$

From this relation, it follows that the maximum current that can flow through the circuit is not affected by R_X or R. On the contrary, it is inversely proportional to η, so the maximum current is achieved for the smallest allowed value $\eta = 0.4$.

Substituting into equation (3), we obtain

$$\frac{U_\mathrm{L}^2}{\eta U\ R_\mathrm{L}} = \frac{U_\mathrm{L}}{R_\mathrm{L}} + \frac{U_\mathrm{L}}{R_\mathrm{X}}\,,$$

$$R_\mathrm{X} = \frac{R_\mathrm{L}}{\frac{1}{\eta}\frac{U_\mathrm{L}}{U} - 1} = 48\,\Omega\,.$$

The portion of the potentiometer with resistance $R - R_\mathrm{X}$ must also carry the current I. The total voltage in the circuit is U, and the voltage across the branched portion is U_L, so according to Kirchhoff's second law, the voltage across this part of the potentiometer is $U' = U - U_\mathrm{L} = 12\,\mathrm{V}$. Using Kirchhoff's first law again, we have

$$\frac{U'}{R - R_\mathrm{X}} = \frac{U_\mathrm{L}}{R_\mathrm{L}} + \frac{U_\mathrm{L}}{R_\mathrm{X}}\,,$$

and thus

$$R = R_\mathrm{X} + \frac{U'}{\frac{U_\mathrm{L}}{R_\mathrm{L}} + \frac{U_\mathrm{L}}{R_\mathrm{X}}} = \frac{\eta U^2}{P_\mathrm{L}}\left(1 + \frac{\eta U_\mathrm{L}}{U_\mathrm{L} - \eta U}\right) = 57.6\,\Omega \doteq 58\,\Omega\,.$$

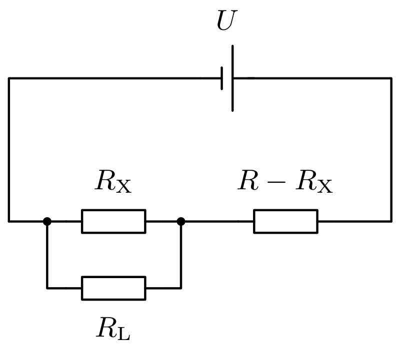

Figure 4: Circuit diagram of the lamp connected via a potentiometer.

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Problem FH ... ice bubble

David was looking perpendicularly at a soap bubble (for simplicity, assume it is made only of water), which appeared green to him due to interference, with a wavelength of $\lambda_0 = 550\,\mathrm{nm}$. *However, since it was really cold outside, the bubble began to freeze. With what wavelength will David see the frozen bubble? Assume that the inner diameter of the bubble remains constant at* $2r = 10.0\,\mathrm{cm}$, *and that David observes only the first order of interference. The refractive index of ice is* $n_\mathrm{l} = 1.31$.

David created an Instagram account and found a video of a bubble freezing.

First, we determine the thickness of the bubble from thin-film interference. If we want an interference maximum, we require that an integer multiple m (called the order of the interference maximum) of the wavelength to be equal to the optical path difference. However, since we are interested in reflection, this must be a half-integer multiple of λ_0 due to a phase change of π upon reflection. In general, the ray may propagate through the layer at some angle θ. For constructive interference, the following relation holds:

$$\lambda_0 \left(m + \frac{1}{2}\right) = 2dn\cos\theta\,,$$

where d is the thickness of the water layer, n its refractive index, and θ is the angle of the ray relative to the normal at the point where David observes the interference. Since David looks at the bubble perpendicularly, $\cos\theta = 1$, and the formula reduces to

$$\lambda_0 \left(m + \frac{1}{2}\right) = 2dn \quad \Rightarrow \quad d = \frac{\lambda_0}{2n}\left(m + \frac{1}{2}\right)\,.$$

Now we must determine how the thickness d changes when water undergoes a phase change. For the volume of a spherical shell, we have

$$V = \frac{4}{3}\pi\left[(R^3) - r^3\right]\,,$$

where r is the inner radius and R is the outer radius. Furthermore, the law of conservation of mass must hold

$$V\rho = V_l\rho_l\,,$$

$$\frac{4}{3}\pi\left[(r+d)^3 - r^3\right]\rho = \frac{4}{3}\pi\left[(r+d_l)^3 - r^3\right]\rho_l\,,$$

$$\left((r+d)^3 - r^3\right)\rho = \left((r+d_l)^3 - r^3\right)\rho_l\,,$$

$$\left(3r^2d + 3rd^2 + d^3\right)\rho = \left(3r^2d_l + 3rd_l^2 + d_l^3\right)\rho_l\,.$$

From the problem statement, we know that $r = 5\,\mathrm{cm}$, and therefore we use the approximation $3rd^2 + d^3 \approx 0$. We thus simplify the equation to

$$3r^2d\rho = 3r^2d_l\rho_l \quad \Rightarrow \quad d_l = d\frac{\rho}{\rho_l}\,,$$

which we substitute back into the second equation to obtain

$$\lambda\left(m_l + \frac{1}{2}\right) = 2d\frac{\rho}{\rho_l}n_l\,.$$

Next, we use the relation between d and λ_0

$$\lambda\left(m_l + \frac{1}{2}\right) = 2\frac{\lambda_0}{2n}\left(m + \frac{1}{2}\right)\frac{\rho}{\rho_l}n_l\,.$$

Finally, using the fact that David observes the same interference order $m = m_l = 1$, we obtain

$$\lambda = \lambda_0\frac{n_l}{n}\frac{\rho}{\rho_l} \doteq 588\,\mathrm{nm}\,.$$

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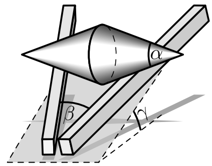
Problem GA ... antigravitational

Marek has two cones of height h and opening angle α glued together at their bases. He places them horizontally midway between two long rods that enclose an angle β and lie in a plane inclined at an angle γ. Both rods have the same inclination with respect to the horizontal plane. Marek watches in surprise as the points of contact between the cones and the rods move upward. What is the smallest possible value of the angle α?



Marek considered the law of gravity to be too down-to-earth.

During the upward motion, "the entire body rises", because the points of contact between the cones and the rods move upward; however, at the same time, the points of contact move away from the base, causing the cone (more precisely, its center of mass) to move downward. Even though the points of contact rise, the center of mass must, overall, descend. Therefore, the second effect must be stronger than the first one. Since we are looking for an extreme (limiting) value, we restrict ourselves to the case in which the two effects are equally strong.

For a horizontal displacement by a distance x, the points of contact between the cone and the rods ascend by $h_{\text{ascent}} = x\tan(\gamma)$. At the same time, during such a displacement the cone moves in the plane of the rods by $x\sec(\gamma)$[4], and the gap therefore widens by $2x\sec(\gamma)\tan(\beta/2)$. In our case, if we increase the separation between the points of contact by r, its center of mass descends by $r\tan(\alpha/2)/2$, because in the limiting case the center of mass and the points of contact lie in the same vertical plane. Altogether, for a horizontal displacement by x, the center of mass descends by

$$h_{\text{descent}} = x\frac{\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\alpha}{2}\right)}{\cos(\gamma)}\,.$$

For the cones to be able to move upward, the limiting case must satisfy

$$h_{\text{ascent}} = h_{\text{descent}}\,,$$

$$\tan\frac{\alpha}{2} = \frac{\sin(\gamma)}{\tan\left(\frac{\beta}{2}\right)}\,.$$

Thus, for the angle α it must hold that

$$\alpha = 2\arctan\left(\frac{\sin(\gamma)}{\tan\left(\frac{\beta}{2}\right)}\right)\,.$$

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[4] The function $\sec(x) = 1/\cos(x)$.

Problem GB ... opaque glass

Let us consider a piece of glass of thickness d, which contains a dark dye that absorbs a portion of the transmitted light. Suppose that the absorption coefficient depends linearly on the dye concentration in the glass as $\mu = \alpha w$, where w is the concentration. If the glass, due to a manufacturing defect, has the standard concentration of dye at the surface, but then the concentration increases linearly by $\Delta w = \beta x$, where x is the depth measured from the surface, by what factor is the transmitted light reduced?

Petr was thinking whether he is going to be able to see anything at all.

We know that defect-free glass has a constant absorption coefficient $\mu = \alpha w$. We can therefore use the empirical Beer-Lambert law, which states that the light intensity decreases exponentially when passing through a homogeneous material as

$$I(x) = I_0 \mathrm{e}^{-\mu x} = I_0 \mathrm{e}^{-\alpha w x}\,.$$

In the case of a non-homogeneous material (i.e., varying dye concentration in the defective glass), we need a more detailed approach. The Beer-Lambert law can be expressed in differential form as

$$\mathrm{d}I(x) = -I\mu(x)\,\mathrm{d}x\,,$$

and since μ depends linearly on the concentration, in our case we have

$$\mathrm{d}I(x) = -I\left(\alpha w + \alpha\beta x\right)\mathrm{d}x\,.$$

This is a simple differential equation that can be solved by separation of variables.

$$\frac{\mathrm{d}I}{I} = -\left(\alpha w + \alpha\beta x\right)\mathrm{d}x \quad\Rightarrow\quad \ln I = -\int\left(\alpha w + \alpha\beta x\right)\mathrm{d}x$$

$$\ln I = -\left(\alpha w x + \frac{\alpha\beta}{2}x^2\right) + C$$

$$I = I_0 \exp\left(-\left(\alpha w x + \frac{\alpha\beta}{2}x^2\right)\right)$$

If we denote the intensity transmitted through the entire defect-free plate as I and the intensity transmitted through the entire defective plate as I' (assuming the same initial intensity), the ratio is

$$\frac{I}{I'} = \exp\left(\frac{\alpha\beta}{2}d^2\right)\,.$$

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Problem GC ... then measure it

Pepa has always wanted to live as a 2D being on a disk. After a lot of time and effort, he has finally achieved it. To celebrate his success, he has decided to measure the radius of the disk on which he is living using a 2D aluminum ruler lying in the same plane as the disk.

However, Pepa has a nefarious friend Vojta who is envious of his success and therefore placed his world on a stove in such a way that the temperature of the disk decreases exponentially with *distance from the center according to the formula $t(r) = t_\mathrm{p} + A\mathrm{e}^{-kr}$, from $t_\mathrm{max} = 160\,^\circ\mathrm{C}$ in the center to $t_\mathrm{min} = 60\,^\circ\mathrm{C}$ at the edge of the disk, where $t_\mathrm{p} = 20\,^\circ\mathrm{C}$ denotes the room temperature.*

How much shorter (in percent) will the radius measured from Pepa's point of view be compared to the true radius of the disk measured by Vojta? Assume that Vojta has a perfectly rigid ruler, Pepa's ruler has a constant coefficient of thermal expansion $\alpha = 2.4\cdot 10^{-5}\,\mathrm{K}^{-1}$ and correctly measures distance at the room temperature t_p. *Pepa was desperate.*

First, let us imagine that we are trying to measure a length at a place with temperature t using an aluminium ruler with the smallest measurable length division x_0. If this temperature were equal to the room temperature t_p, it would be sufficient to simply count the number of elementary divisions of the ruler, N, covered by the measured segment. From this, we could determine the length of the segment as Nx_0. However, if the temperature at the given location differs from the room temperature, the entire ruler, as well as its elementary division, will expand (or contract, respectively) to $(1 + \alpha(t - t_\mathrm{p}))$ times the original length. Instead of N elementary divisions, the measured segment will then cover $N/(1 + \alpha(t - t_\mathrm{p}))$ elementary divisions. To this number we assign the length $Nx_0/(1 + \alpha(t - t_\mathrm{p}))$ instead of the length Nx_0, meaning that we measure a length that is $(1 + \alpha(t - t_\mathrm{p}))$ times smaller than the actual one.

Let us now return to the situation from the problem statement. We measure the radius of a disk whose true radius is R. Its temperature decreases exponentially with radius according to the relation $t(r) = t_\mathrm{p} + A\mathrm{e}^{-kr}$, from the temperature t_max to the temperature t_min. Let us first determine the values of the unknown constants A and k from the given information. For $r = 0$, we must have $t = t_\mathrm{max}$, so after substituting into the temperature expression we obtain

$$t_\mathrm{max} = t_\mathrm{p} + A\,,$$

from which we can easily express the constant A as $A = t_\mathrm{max} - t_\mathrm{p}$.

At the edge of the disk, on the other hand, we have $r = R$ and $t = t_\mathrm{min}$, so

$$t_\mathrm{min} = t_\mathrm{p} + A\mathrm{e}^{-kR}\,.$$

After substituting $A = t_\mathrm{max} - t_\mathrm{p}$ and rearranging, we successively obtain

$$\frac{t_\mathrm{min} - t_\mathrm{p}}{t_\mathrm{max} - t_\mathrm{p}} = \mathrm{e}^{-kR}\,,$$

$$k = -\frac{1}{R}\ln\frac{t_\mathrm{min} - t_\mathrm{p}}{t_\mathrm{max} - t_\mathrm{p}}\,.$$

The dependence of temperature on radius thus becomes

$$t(r) = t_\mathrm{p} + (t_\mathrm{max} - t_\mathrm{p})\left(\frac{t_\mathrm{min} - t_\mathrm{p}}{t_\mathrm{max} - t_\mathrm{p}}\right)^{r/R}\,.$$

Now imagine measuring the radius of this disk from its centre to the edge using an aluminium ruler. The length of an elementary radial segment with true length $\mathrm{d}r$, which we measure at a distance r from the centre of the disk, will be

$$\mathrm{d}r' = \frac{\mathrm{d}r}{1 + \alpha\left(t_\mathrm{max} - t_\mathrm{p}\right)\left(\frac{t_\mathrm{min} - t_\mathrm{p}}{t_\mathrm{max} - t_\mathrm{p}}\right)^{r/R}}\,.$$

The radius measured by Pepa will then be

$$R' = \int_0^R \frac{\mathrm{d}r}{1 + \alpha\,(t_{\max} - t_{\mathrm{p}}) \left(\frac{t_{\min} - t_{\mathrm{p}}}{t_{\max} - t_{\mathrm{p}}}\right)^{r/R}} .$$

For simplicity, we introduce the substitutions $\beta = \alpha\,(t_{\max} - t_{\mathrm{p}})$, $B = (t_{\min} - t_{\mathrm{p}})/(t_{\max} - t_{\mathrm{p}})$ and $x = r/R$. After additionally expressing $\mathrm{d}r = R\,\mathrm{d}x$, the integral takes the form

$$R' = R \int_0^1 \frac{1}{1 + \beta B^x}\,\mathrm{d}x .$$

If we now cleverly rewrite unity in the numerator as $1 + \beta B^x - \beta B^x$, the expression becomes

$$R' = R \int_0^1 \left(1 - \frac{\beta B^x}{1 + \beta B^x}\right) \mathrm{d}x = R - R \int_0^1 \frac{\beta B^x}{1 + \beta B^x} .$$

After the substitution $u = 1 + \beta B^x$, from which we obtain $\mathrm{d}u = \beta B^x \ln B\,\mathrm{d}x$, the term βB^x in the numerator cancels out and we obtain

$$R' = R - \frac{R}{\ln B} \int_{1+\beta}^{1+\beta B} \frac{\mathrm{d}u}{u} = R - \frac{R}{\ln B} \ln \frac{1 + \beta B}{1 + \beta} .$$

Compared to Vojta, Pepa measures a radius smaller by

$$\frac{R - R'}{R} = \ln \frac{1 + \alpha\,(t_{\min} - t_{\mathrm{p}})}{1 + \alpha\,(t_{\max} - t_{\mathrm{p}})} / \ln \frac{t_{\min} - t_{\mathrm{p}}}{t_{\max} - t_{\mathrm{p}}} \doteq 0.19\% .$$

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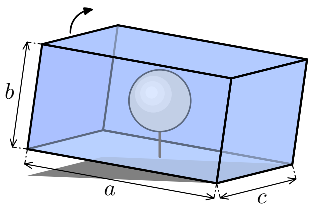
Problem GD ... overturning a rectangular cuboid

Consider a hollow rectangular cuboid with a base of side lengths $a = 50.0\,\mathrm{cm}$ and $c = 30.0\,\mathrm{cm}$ and height $b = 30.0\,\mathrm{cm}$. The walls of the cuboid have negligible mass, but its entire volume is filled with water and a spherical buoy of radius $r = 8.00\,\mathrm{cm}$ made of a material with density $\rho_0 = 350\,\mathrm{kg{\cdot}m^{-3}}$. The buoy is attached to the center of the cuboid base by a massless and volumeless string of length $l = a/2 - 2r$. What work is needed to tip the cuboid over the edge c? Assume that the tipping is performed very slowly.



Petr's cuboid went crashing down.

First, we choose a suitable coordinate system. It turns out to be advantageous to choose it so that the origin is on the edge about which we rotate the rectangular cuboid; the x- and y-axes point in the horizontal and vertical directions, respectively, with the positive direction of the x-axis pointing toward the cuboid. It suffices to solve the problem in two dimensions; therefore, we do not have to consider the z-axis. Now we determine the coordinates of the center of mass. In general, for the j-th coordinate of the center of mass of a system, we have

$$t_j = \frac{\sum_i m_i x_{ij}}{\sum_i m_i}\,,$$

where m_i are the masses of the bodies composing the system and x_{ij} is the j-th coordinate of the center of mass of the i-th body. In other words, the center of mass is the weighted average of the coordinates of the centers of mass of the individual bodies, where the weights are the masses of these bodies. Thus, for the x- and y-coordinates of the center of mass of our system, we obtain

$$t_x = \frac{\frac{1}{2}a^2bc\rho - \frac{2}{3}\pi r^3 a\left(\rho-\rho_0\right)}{abc\rho - \frac{4}{3}\pi r^3\left(\rho-\rho_0\right)}\,,$$

$$t_y = \frac{\frac{1}{2}ab^2c\rho - \frac{4}{3}\pi r^3\left(\frac{a}{2}-r\right)\left(\rho-\rho_0\right)}{abc\rho - \frac{4}{3}\pi r^3\left(\rho-\rho_0\right)}\,.$$

We do not know exactly where the center of mass of the water in the cuboid is located, so we use the following trick: we consider the cuboid filled with water without the buoy, which has its center of mass (as we would expect) in the center of the cuboid, and to it we add a body of the same shape and position as the buoy with a (purely formal) negative density $-\rho$. Then, it was sufficient to add the actual buoy with density ρ_0.

Let us now imagine the cuboid as we begin to tilt it. Due to the buoyant force, the buoy is always pulled upward. Let us parametrize all positions by the tilt angle of the cuboid with respect to the surface φ. With a bit of geometry, we obtain the position of the center of mass in this configuration

$$t'_x = \frac{\frac{1}{2}\left(a\cos\varphi - b\sin\varphi\right)abc\rho - \frac{2}{3}\pi r^3 a\cos\varphi\left(\rho-\rho_0\right)}{abc\rho - \frac{4}{3}\pi r^3\left(\rho-\rho_0\right)}\,,$$

$$t'_y = \frac{\frac{1}{2}\left(a\sin\varphi + b\cos\varphi\right)abc\rho - \frac{4}{3}\pi r^3\left(\frac{a}{2}+\frac{a}{2}\sin\varphi - r\right)\left(\rho-\rho_0\right)}{abc\rho - \frac{4}{3}\pi r^3\left(\rho-\rho_0\right)}\,.$$

We now determine the limiting angle φ_M. We can determine it in two ways, either using the derivative of t'_y with respect to φ or from the condition $t'_x = 0$. In both cases, we obtain

$$\varphi_\mathrm{M} = \arctan\left(\frac{a}{b} - \frac{4}{3}\pi r^3\frac{1}{b^2c}\left(1-\frac{\rho_0}{\rho}\right)\right) \doteq 58.2^\circ\,.$$

The work is calculated as

$$W = \Delta E = g\left(abc\rho - \frac{4}{3}\pi r^3\left(\rho-\rho_0\right)\right)\left(t'_y\left(\varphi_\mathrm{M}\right) - t_y\right)\,,$$

which can be further simplified to

$$W = \frac{a\rho g}{2}\left(\sqrt{\left(abc - \frac{4}{3}\pi r^3\left(1-\frac{\rho_0}{\rho}\right)\right)^2 + \left(b^2c\right)^2} - b^2c\right) \doteq 59.5\,\mathrm{J}\,.$$

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Problem GE ... networking

Consider a so-called multimode optical fiber in which an LED serves as the light source from which information propagates. The diode is located at the center of a tube with diameter $D = 62.5\,\mu\text{m}$ *and refractive index* $n = 1.48$, *and it emits light isotropically (equally in all directions). For simplicity, assume that the tube is surrounded by air and that the entire optical link has a length of* $l = 1.00\,\text{km}$. *Calculate the mean value of the arrival times of light rays that were emitted at the same instant at the end of the cable. Consider only those light rays that reach the destination.* *During a lecture on computer networks, Vlado was wondering why multimode cables are used only over relatively short distances.*

Let us consider a single ray emitted by the LED at an angle φ with respect to the axis of the cable. Let us first derive the condition under which the ray does not leave the cable at its boundary, but is instead only reflected. Upon incidence on the interface, the ray forms an angle $\pi/2 - \varphi$ with the normal. Therefore, for total internal reflection to occur, the following condition must hold

$$n \sin\left(\frac{\pi}{2} - \varphi\right) = n\cos\varphi > n_0 \sin\frac{\pi}{2} \approx 1\,,$$

where we have used the trigonometric identity $\sin(\pi/2 - x) = \cos x$ as well as the refractive index of air $n_0 \approx 1$. After rearranging and taking into account that the cosine function is decreasing on the interval $\langle 0, \pi/2\rangle$, we obtain the following condition for the angle φ:

$$\varphi < \arccos\left(\frac{1}{n}\right) = \varphi_{\text{m}} \doteq 47.5^\circ \tag{4}$$

Rays emitted at larger angles φ_{m} will also be partially reflected to some extent. However, their intensity decreases with each reflection, and thus their contribution to the signal received at the far end of the cable becomes negligible at large distances. For this reason, we do not consider them in the remainder of the solution.

If the condition (4) is satisfied, the ray will form the same angle φ with the cable axis after each reflection. As a result, total internal reflection will also occur at subsequent reflections, since the ray will always make the same angle with the interface between the cable and the air. The information propagates through the cable with refractive index n at a speed $v = c/n$. Since the ray maintains a constant angle φ with the cable axis, it must, in fact, travel a distance $x = l/\cos\varphi$ instead of the cable length l. The information, therefore, reaches the other end of the cable after a time

$$t(\varphi) = \frac{x}{v} = \frac{nl}{c\cos\varphi}\,.$$

To determine the mean value of this time, we must also find what fraction of all rays satisfying the condition (4) is emitted within a small angular interval $\langle \varphi, \varphi + \mathrm{d}\varphi)$. Let us imagine a fictitious sphere of radius R centered at the LED, onto whose surface the rays emitted isotropically from the diode fall uniformly. It then suffices to determine the ratio of the area of the part of the sphere corresponding to this angular interval to the area of the part of the sphere corresponding to the entire admissible angular interval $\langle 0, \varphi_{\text{m}})$. The part of the sphere corresponding to the small angular interval $\langle \varphi, \varphi + \mathrm{d}\varphi)$ forms a thin spherical ring with inner radius $R\sin\varphi$ and thickness $R\,\mathrm{d}\varphi$, and thus with area

$$\mathrm{d}S = 2\pi R\sin\varphi \cdot R\,\mathrm{d}\varphi = 2\pi R^2 \sin\varphi\,\mathrm{d}\varphi\,.$$

The area of the part of the sphere corresponding to the entire admissible angular interval $\langle 0, \varphi_\mathrm{m})$ is then given by

$$S = \int_0^{\varphi_\mathrm{m}} 2\pi R^2 \sin\varphi \,\mathrm{d}\varphi = 2\pi R^2 \left(1 - \cos\varphi_\mathrm{m}\right)$$

The fraction of rays that fall into the small angular interval $\langle \varphi, \varphi + \mathrm{d}\varphi)$ is therefore

$$\frac{\mathrm{d}S}{S} = \frac{\sin\varphi \,\mathrm{d}\varphi}{1 - \cos\varphi_\mathrm{m}} \,.$$

We can now express the mean value of the time t. If we had only a finite number of admissible angles φ, the mean value of the time t would be a weighted average of the times $t(\varphi)$ corresponding to the individual admissible angles φ, with the weights given by the fractions of rays emitted at those angles. In the continuous case, an analogous relation holds, except that instead of a sum in the weighted average, we must use an integral. At the same time, the fraction of rays must be associated not with a single specific value of φ, but with a small angular interval $\langle \varphi, \varphi + \mathrm{d}\varphi)$, as we have done above. The mean value of the time t in our case is therefore

$$\langle t \rangle = \int_0^{\varphi_\mathrm{m}} \frac{nl}{c\cos\varphi} \frac{\sin\varphi \,\mathrm{d}\varphi}{1 - \cos\varphi_\mathrm{m}} = \frac{nl}{c\left(1 - \cos\varphi_\mathrm{m}\right)} \int_0^{\varphi_\mathrm{m}} \frac{\sin\varphi}{\cos\varphi} \,\mathrm{d}\varphi \,.$$

To evaluate the integral of the tangent, it is convenient to perform the substitution $u = \cos\varphi$, since after expressing $\mathrm{d}\varphi = -\,\mathrm{d}u / \sin\varphi$, the sine cancels from the integral. The expression for the mean value then becomes

$$\langle t \rangle = \frac{-nl}{c\left(1 - \cos\varphi_\mathrm{m}\right)} \int_1^{\cos\varphi_\mathrm{m}} \frac{1}{u} \,\mathrm{d}u = \frac{-nl}{c\left(1 - \cos\varphi_\mathrm{m}\right)} \ln\left(\cos\varphi_\mathrm{m}\right) = \frac{n^2 l}{c(n-1)} \ln(n) \doteq 5.97\,\mathrm{\mu s} \,.$$

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Problem GF ... indistinguishable gases

There is a mixture of two gases in a vacuum chamber at low pressure – nitrogen and carbon monoxide, with a portion of them being ionized. We can identify the gases in a mass spectrometer by the ratio of their charge to mass. However, the relative molecular mass of both gases is very similar, approximately $M = 28$, and our spectrometer does not have sufficient resolution to distinguish them. Some particles are ionized twice, which manifests as a signal at the position $M = 14$. The ratio of the cross section of double ionization to the cross section of single ionization is 0.015 for CO, and 0.090 for N_2. The ratio of the cross section of the first ionization of N_2 to the first ionization of CO is then 0.83. We would like to determine the concentrations of both gases from the intensity of the detected signal at the given positions. The signal is measured as the amplified current of ions that strike the detector. We detected $I_{28} = 210\,\mathrm{\mu A}$ at the position with molecular mass 28, and $I_{14} = 10.5\,\mathrm{\mu A}$ at the position 14. Determine the ratio of the concentrations of carbon monoxide to nitrogen. Do not consider their mutual interactions.

Today at the lecture, today at the problem selection.

In a mass spectrometer, individual particles are ionized so that their trajectories are influenced by electric and magnetic fields in such a way that we can distinguish the individual particles types. Charged particle trajectories are affected by the Lorentz force

$$\mathbf{F} = m\mathbf{a} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) ,$$

where $\mathbf{E}$ is the electric field vector and $\mathbf{B}$ the magnetic induction vector. The acceleration, and hence the particle's trajectory, thus depend on the fields with a proportionality constant e/m. Two particles with the same charge have different trajectories if they have different masses. Therefore, we can separate species based on the ratio m/e. Doubly ionized particles then appear in the spectrum as particles with half the mass.

According to the problem statement, we detect electric currents at the position corresponding to particle mass 28 and at the position 14, which corresponds to double ionization of these molecules. Let us denote the partial pressure of nitrogen by $p_{\mathrm{N_2}}$ and the partial pressure of carbon monoxide by p_{CO}. The first-ionization cross sections is denoted as $\sigma_{1,\mathrm{N_2}}$ for nitrogen and $\sigma_{1,\mathrm{CO}}$ for carbon monoxide. Then the signal intensity at position 28 can be expressed as

$$I_{28} = k\,\left(\sigma_{1,\mathrm{N_2}} p_{\mathrm{N_2}} + \sigma_{1,\mathrm{CO}} p_{\mathrm{CO}}\right) ,$$

where k is the proportionality constant between the number of detected ions and the number of all ions. In our case, the intensity equals the current, because we measure the number of detected charged particles, i.e., essentially the charge that has passed. Similarly, we can express the intensity at position 14 as

$$I_{14} = 2k\left(\sigma_{2,\mathrm{N_2}} p_{\mathrm{N_2}} + \sigma_{2,\mathrm{CO}} p_{\mathrm{CO}}\right) ,$$

where the numeral 2 in front of the entire parenthesis signifies that a single ion now carries twice the charge.

We divide the first equation by the second equation and simplify

$$\frac{I_{28}}{I_{14}} = \frac{1}{2}\frac{\sigma_{1,\mathrm{N_2}} p_{\mathrm{N_2}} + \sigma_{1,\mathrm{CO}} p_{\mathrm{CO}}}{\sigma_{2,\mathrm{N_2}} p_{\mathrm{N_2}} + \sigma_{2,\mathrm{CO}} p_{\mathrm{CO}}} ,$$

$$2\frac{I_{28}}{I_{14}} = \frac{\sigma_{1,\mathrm{N_2}} + \sigma_{1,\mathrm{CO}}\frac{p_{\mathrm{CO}}}{p_{\mathrm{N_2}}}}{\sigma_{2,\mathrm{N_2}} + \sigma_{2,\mathrm{CO}}\frac{p_{\mathrm{CO}}}{p_{\mathrm{N_2}}}} ,$$

$$2\frac{I_{28}}{I_{14}}\sigma_{2,\mathrm{CO}}\frac{p_{\mathrm{CO}}}{p_{\mathrm{N_2}}} - \sigma_{1,\mathrm{CO}}\frac{p_{\mathrm{CO}}}{p_{\mathrm{N_2}}} = \sigma_{1,\mathrm{N_2}} - 2\frac{I_{28}}{I_{14}}\sigma_{2,\mathrm{N_2}} ,$$

$$\frac{p_{\mathrm{CO}}}{p_{\mathrm{N_2}}} = \frac{\sigma_{1,\mathrm{N_2}} - \frac{2I_{28}}{I_{14}}\sigma_{2,\mathrm{N_2}}}{2\frac{I_{28}}{I_{14}}\sigma_{2,\mathrm{CO}} - \sigma_{1,\mathrm{CO}}} ,$$

$$\frac{p_{\mathrm{CO}}}{p_{\mathrm{N_2}}} = \frac{\sigma_{1,\mathrm{N_2}}}{\sigma_{1,\mathrm{CO}}}\,\frac{1 - \frac{2I_{28}}{I_{14}}\frac{\sigma_{2,\mathrm{N_2}}}{\sigma_{1,\mathrm{N_2}}}}{2\frac{I_{28}}{I_{14}}\frac{\sigma_{2,\mathrm{CO}}}{\sigma_{1,\mathrm{CO}}} - 1} .$$

We only need to substitute the ratios given in the problem statement to obtain the result

$$\frac{p_{\mathrm{CO}}}{p_{\mathrm{N}_2}} = 0.83\frac{\frac{2I_{28}}{I_{14}}0.09 - 1}{1 - 2\frac{I_{28}}{I_{14}}0.015} \doteq 5.4\,.$$

Therefore, there is significantly more carbon monoxide than nitrogen in this atmosphere.

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Problem GG ... repulsive light

You have surely experienced walking out of a dimly lit building and suddenly being blinded by the Sun. Calculate the force exerted by the Sun on the Earth due to its radiation. Assume that the Earth's surface is entirely composed of water, which means that a fraction of $\alpha = 0.31$ of all incident radiation is perfectly reflected from the surface, while the rest of it is absorbed.

Vlado left the school and immediately turned back.

The solution to the problem is the sum of two effects — in the first, we consider that a fraction $1-\alpha$ of the photons is absorbed, and in the second, that a fraction α of the photons is reflected from Earth as from a mirror.

Since Earth is rotationally symmetric about the axis given by the Sun–Earth line, we can parametrize the problem by the angle φ, which the line connecting Earth's center and a point on its surface makes with the Sun–Earth line. These points have the property that sunlight shines on these locations on Earth at the same angle, namely φ from the surface normal. The region on the surface of the sphere located between angles φ and $\varphi + \mathrm{d}\varphi$ corresponds to an area element $\mathrm{d}A = 2\pi R_{\oplus} \sin\varphi \cdot R_{\oplus}\,\mathrm{d}\varphi$.

We can model the interaction of photons with Earth as collisions. In the first case, a fraction $1-\alpha$ of the photons is inelastically absorbed by Earth; therefore, their momentum changes by $\Delta\mathbf{p}_1 = 0 - \mathbf{p} = -\mathbf{p}$. In the second case, an elastic collision happens according to the law of reflection, according to which only the momentum component $\mathbf{p}_{\perp}$, perpendicular to the surface at the point of reflection, changes. The momentum of α photons therefore changes by $\Delta\mathbf{p}_2 = \left(\mathbf{p}_{\parallel} - \mathbf{p}_{\perp}\right) - \left(\mathbf{p}_{\parallel} + \mathbf{p}_{\perp}\right) = -2\mathbf{p}_{\perp}$.

Next, we express the photon momentum in terms of the power P with which the Sun radiates toward Earth at the Sun-Earth distance. For the momentum of N photons that strike Earth during a time interval Δt and have wavelength λ, the following holds

$$p = N\frac{h}{\lambda} = N\frac{h}{\frac{hc}{E}} = \frac{NE}{c} = \frac{P\Delta t}{c}\,,$$

where the relation for the photon energy $E = \frac{hc}{\lambda}$ was used.

The force with which the radiation acts on Earth can be calculated using Newton's second law,

$$\mathbf{F} = \frac{\Delta\mathbf{p}}{\Delta t} = -\left((1-\alpha)\frac{\Delta\mathbf{p}_1}{\Delta t} + \alpha\frac{\Delta\mathbf{p}_2}{\Delta t}\right)\,.$$

According to the law of conservation of momentum, the change in Earth's momentum is opposite in direction to the change in the photons' momentum, so the minus sign was used in the equation above. As a consequence, Earth is "pushed away" from the Sun by the radiation.

Due to the symmetry of the situation, the force acts along the Sun–Earth line, which we will hereafter call the "parallel direction". With respect to the resulting force $\mathbf{F}$, we will consider only the change of momentum in the parallel direction ($F_{\parallel}$). In practice, however, it is convenient to introduce the radiation pressure instead of the force:

$$\mathcal{P} = \frac{F}{A} = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{P}{A}\frac{1}{c} =: \frac{I}{c}\,,$$

where I is the power carried by the radiation, per unit area oriented perpendicular to the rays, at Earth's distance from the Sun.

Let us consider the first case, in which all solar rays are absorbed. Since the photons propagate in the parallel direction, their momentum change $\Delta p_1 = p$ is in the same direction. The pressure $\mathcal{P}$ in this case means that an element of force $\mathrm{d}F_{\parallel}$ acts on an area element $\mathrm{d}A_{\mathrm{eff}}$ that is perpendicular to the rays. The area element $\mathrm{d}A$ makes an angle φ with the rays. Therefore, we obtain $\mathrm{d}A_{\mathrm{eff}}$ by projecting $\mathrm{d}A$ into the direction perpendicular to the rays, which gives

$$\mathrm{d}A_{\mathrm{eff}} = \mathrm{d}A\cos\varphi = 2\pi R^2 \sin\varphi\cos\varphi\,\mathrm{d}\varphi\,.$$

Thus, the resulting force acting on Earth in the parallel direction is

$$\begin{aligned} F_{1\parallel} &= \int_{\text{hemisphere}} \mathcal{P}\,\mathrm{d}A_{\mathrm{eff}} = \mathcal{P}\cdot 2\pi R_{\oplus}^2 \int_0^{\pi/2} \sin\varphi\cos\varphi\,\mathrm{d}\varphi = 2\pi\mathcal{P}R_{\oplus}^2 \int_0^{\pi/2} \frac{\sin(2\varphi)}{2}\,\mathrm{d}\varphi \\ &= 2\pi\mathcal{P}R_{\oplus}^2\cdot\frac{1}{2} = \mathcal{P}\cdot\pi R_{\oplus}^2\,. \end{aligned}$$

We obtained the well-known result that, if parallel rays illuminate a sphere, then the effective surface area of Earth in the perpendicular direction is $\pi R_{\oplus}^2$.

Let us now analyze the second, more complicated case. As indicated in the introduction, the momentum change $\Delta\mathbf{p}_2$ is perpendicular to the surface; therefore, it points in the radial direction outward from Earth's center at the given point of reflection (the situation is analogous to the incidence of rays on a plane at an angle φ from the normal). The magnitude of $\Delta\mathbf{p}_2$ equals twice the momentum component perpendicular to the surface; that is, $\Delta p_2 = 2p_{\perp} = 2p\cos\varphi$. From Δp_2, however, we are interested only in the component parallel to the Sun–Earth line, namely

$$\Delta p_{2\parallel} = \Delta p_2\cos\varphi = 2p\cos^2\varphi = p\left(1+\cos\left(2\varphi\right)\right)\,.$$

Analogously to the first case, we must again take into account the effective area $\mathrm{d}A_{\mathrm{eff}}$ instead of the total area $\mathrm{d}A$. The resulting force is then

$$\begin{aligned} F_{2\parallel} &= \int_{\text{hemisphere}} \mathcal{P}\left(1+\cos\left(2\varphi\right)\right)\mathrm{d}A_{\mathrm{eff}} = \mathcal{P}\cdot 2\pi R_{\oplus}^2 \int_0^{\pi/2} \left(\sin\varphi\cos\varphi + \sin\varphi\cos\varphi\cos(2\varphi)\right)\mathrm{d}\varphi \\ &= 2\pi\mathcal{P}R_{\oplus}^2\left(\int_0^{\pi/2}\frac{\sin(2\varphi)}{2}\,\mathrm{d}\varphi + \int_0^{\pi/2}\frac{\sin(4\varphi)}{4}\,\mathrm{d}\varphi\right) = 2\mathcal{P}\pi R_{\oplus}^2\cdot\left(\frac{1}{2}+0\right) = \mathcal{P}\cdot\pi R_{\oplus}^2\,. \end{aligned}$$

We obtained a surprising result — the effective surface area for complete reflection is the same as for complete absorption. It follows that, in fact, the resulting force acting on Earth does not depend on the albedo at all.

At this point, it remains only to determine the value of the intensity I. The Sun radiates with luminosity $L_{\odot}$. This light propagates from the Sun equally in all directions; therefore, at a distance $r = 1\,\mathrm{au}$, the power per unit area is

$$I = \frac{L_{\odot}}{4\pi r^2}\,.$$

Finally, we obtain

$$F_{\parallel} = (1-\alpha)F_{1\parallel} + \alpha F_{2\parallel} = \frac{I}{c}\pi R_{\oplus}^2 = \frac{L_{\odot}}{4c}\frac{R_{\oplus}^2}{r^2} \doteq 5.8\cdot 10^8\,\mathrm{N}\,.$$

This force is much smaller than the gravitational force with which the Sun acts on Earth, whose magnitude is $\sim 3.5\cdot 10^{22}\,\mathrm{N}$.

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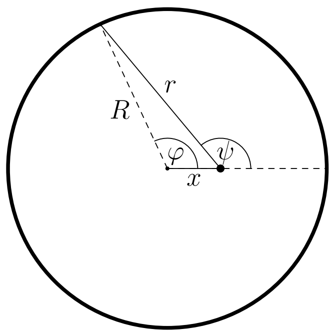
Problem GH ... charged ring

Kubo tried to create a trap for charged particles. For this purpose, he took a thin, uniformly electrically charged ring with radius $R = 1.0\,\mathrm{cm}$ and linear charge density $\lambda = 9.0\cdot 10^{-6}\,\mathrm{C\cdot m^{-1}}$. He then placed a charged particle with specific charge $q/m = 5.2\cdot 10^8\,\mathrm{C\cdot kg^{-1}}$ at its center. In the direction perpendicular to the plane of the ring, this was unfortunately only an unstable equilibrium position, so he slightly displaced the particle from the center only within the ring's plane. Determine the period of the particle's initial motion around the center of the ring.

Kubo wished to analytically solve the integral from the problem "Faraday's collector".



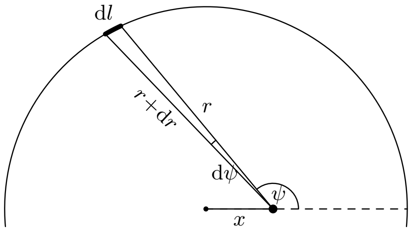
To determine the period of small oscillations, we must first determine the resultant force acting on the charged particle due to the presence of the charged ring. The ring acts only via Coulomb forces, but from each of its points. Let us denote the displacement of the charged particle from the center of the ring by x, and the distance of an element of the ring (in the direction of ψ) from the particle by r. This distance can be calculated using the law of cosines as $r^2 = R^2 + x^2 - 2Rx\cos\varphi$, where φ is the angular coordinate of the element on the ring with respect to its center. This angle can be determined using the law of sines for the sides R and x.

$$\frac{R}{\sin(\pi-\psi)} = \frac{x}{\sin(\psi-\varphi)} \quad\Rightarrow\quad \varphi = \psi - \arcsin\left(\frac{x}{R}\sin\psi\right)\,,$$

$$\cos\varphi = \cos(\psi)\cos\left(\arcsin\left(\frac{x}{R}\sin\psi\right)\right) + \sin(\psi)\frac{x}{R}\sin\psi = \sqrt{1-\frac{x^2}{R^2}\sin^2\psi}\cos\psi + \frac{x}{R}\sin^2\psi\,.$$

By substituting this into the law of cosines, we obtain the distance r as a function of the angle ψ. For small values of x, we can simplify the expression by neglecting terms of order $\mathcal{O}(x^2)$.

$$r^2 = R^2 + x^2 - 2Rx\sqrt{1-\frac{x^2}{R^2}\sin^2\psi}\cos\psi - 2x^2\sin^2\psi \approx R^2 - 2Rx\cos\psi,.$$



Next, we still need to express the length element of the ring $\mathrm{d}l$ in terms of the differential $\mathrm{d}\psi$. We again compute this using the law of cosines.

$$\mathrm{d}l^2 = r^2 + \left(r + \frac{\mathrm{d}r}{\mathrm{d}\psi}\,\mathrm{d}\psi\right)^2 - 2r\left(r + \frac{\mathrm{d}r}{\mathrm{d}\psi}\,\mathrm{d}\psi\right)\cos\mathrm{d}\psi = \left(\frac{\mathrm{d}r}{\mathrm{d}\psi}\,\mathrm{d}\psi\right)^2 + r^2\,\mathrm{d}\psi^2 + \mathcal{O}\left(\mathrm{d}\psi^3\right)$$

$$\mathrm{d}l = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\psi}\right)^2}\,\mathrm{d}\psi = \sqrt{r^2 + \mathcal{O}(x^2)}\,\mathrm{d}\psi \approx r\,\mathrm{d}\psi$$

The derivative of the function $r(\psi)$ is of order $\mathcal{O}(x)$, and its square therefore behaves at least as $\mathcal{O}(x^2)$, so it can be neglected in comparison with the value of r^2.

Now, nothing prevents us from calculating the total electric field acting on the charged particle. Normally, we could compute it through vectors in 3D, but here we can exploit the axial symmetry of the problem and restrict ourselves to the component of the field intensity in the direction of the displacement x. We therefore include a prefactor $\cos\psi$.

$$|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0}\oint\frac{\lambda\cos\psi}{r^2}\,\mathrm{d}l \approx \frac{\lambda}{2\pi\varepsilon_0}\int_0^{\pi}\frac{\cos\psi}{r}\,\mathrm{d}\psi \approx \frac{\lambda}{2\pi\varepsilon_0}\int_0^{\pi}\frac{\cos\psi}{R}\left(1 + \frac{x}{R}\cos\psi\right)\mathrm{d}\psi =$$

$$= \frac{\lambda x}{2\pi R^2\varepsilon_0}\int_0^{\pi}\cos^2\psi\,\mathrm{d}\psi = \frac{\lambda x}{2\pi R^2\varepsilon_0}\int_0^{\pi}\frac{1+\cos(2\psi)}{2}\,\mathrm{d}\psi = \frac{\lambda x}{2\pi R^2\varepsilon_0}\left[\frac{\psi}{2}\right]_0^{\pi} = \frac{\lambda x}{4R^2\varepsilon_0}\,.$$

In the calculation, we used the approximation $r^{-1} \approx R^{-1}\left(1 + (x/R)\cos\psi\right)$, which follows from the expansion of $r(x)$ and is valid for small values of x. In evaluating the final integral, we also used the fact that the integral of cosine from 0 to π, as well as from 0 to 2π, is zero.

Finally, we only need to write down the equation of motion of the charged particle with charge q and mass m, identify the equation of a harmonic oscillator, and determine the period of small oscillations.

$$ma = -\frac{q\lambda x}{4R^2\varepsilon_0} = -m\omega^2 x \quad\Rightarrow\quad \omega^2 = \frac{q\lambda}{4mR^2\varepsilon_0}\,,$$

$$T = \frac{2\pi}{\omega} = 4\pi R\sqrt{\frac{m\varepsilon_0}{q\lambda}} \doteq 5.47\cdot 10^{-9}\,\mathrm{s}\,.$$

The sought period of small oscillations of the charged particle is approximately $T \doteq 5.5\,\mathrm{ns}$.

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Problem HA ... friction on an inclined plane

We have a point mass on an inclined plane with a variable slope in a homogeneous gravitational field. The angle α between the plane and the horizontal plane is slowly increased to a value α_1, at which the point mass begins to move. Subsequently, the angle α is continuously decreasing with an angular velocity $\omega = 1\,^\circ\cdot\mathrm{s}^{-1}$ in such a way that the axis of rotation of the plane passes through the point mass. What distance along the inclined plane does the point mass travel between its two stationary positions? The coefficient of static friction is $f_\mathrm{s} = 0.65$ and the coefficient of dynamic friction is $f_\mathrm{d} = 0.51$. *Dávid enrolled in a bachelor's revision course.*

The problem is solvable in two dimensions, so we introduce a coordinate system where the x-axis is parallel to the inclined plane and the y-axis is perpendicular to it. By decomposing the acting forces into these two directions, we obtain the following system of equations:

$$
\begin{aligned}
x: &\quad \mathbf{T} + \mathbf{G} = m\mathbf{a} &\Rightarrow&\quad mg\sin\alpha - T = ma\,,\\
y: &\quad \mathbf{N} + \mathbf{G} = \mathbf{0} &\Rightarrow&\quad mg\cos\alpha - N = 0\,,
\end{aligned}
$$

where T denotes the friction force, G the gravitational force, N the normal force exerted on the point mass by the surface, m its mass, g the gravitational acceleration, and finally a denotes the resulting acceleration with which the point mass moves.

At time $t = 0$, the angle α is exactly such that the forces in the x-direction balance out, from which we obtain an equation for calculating the angle α_1

$$
mg\sin(\alpha_1) = f_\mathrm{s} mg\cos(\alpha_1) \quad\Rightarrow\quad \alpha_1 = \arctan(f_\mathrm{s})\ .
$$

We know that as soon as the point mass starts to move, the angle α also begins to change at a constant rate of ω. Therefore, $\alpha(t) = \alpha(0) - \omega t = \alpha_1 - \omega t$. At a general time t when the point mass is moving, we obtain from Newton's second law

$$
mg\sin\alpha(t) - f_\mathrm{d} mg\cos\alpha(t) = ma \quad\Rightarrow\quad a = g(\sin(\alpha_1 - \omega t) - f_\mathrm{d}\cos(\alpha_1 - \omega t))\,.
$$

By integrating the last equation with respect to time, we obtain the relation for $v(t)$

$$
v(t) = \frac{g}{\omega}\Big(\cos(\alpha_1 - \omega t) + f_\mathrm{d}\sin(\alpha_1 - \omega t)\Big) + C_1\,.
$$

Using the condition that at time $t = 0$ the velocity is zero, we eliminate the integration constant C_1 and obtain

$$
v(t) = \frac{g}{\omega}\Big(\cos(\alpha_1 - \omega t) - \cos(\alpha_1) + f_\mathrm{d}(\sin(\alpha_1 - \omega t) - \sin(\alpha_1))\Big)\,.
$$

From this equation, we can now calculate the time t_max at which the point mass comes to rest again, i.e., when $v(t_\mathrm{max}) = 0$. To isolate the time t in the equation, we first use the sum formulas for trigonometric functions, $\sin(\alpha_1 - \omega t) = \sin(\alpha_1)\cos(\omega t) - \cos(\alpha_1)\sin(\omega t)$ and

$\cos(\alpha_1 - \omega t) = \cos(\alpha_1)\cos(\omega t) + \sin(\alpha_1)\sin(\omega t)$. By substituting these and rearranging the previous equation, we obtain

$$v(t) = \frac{g}{\omega}\Big((\cos(\alpha_1) + f_\mathrm{d}\sin(\alpha_1))\,(\cos(\omega t) - 1) + (\sin(\alpha_1) - f_\mathrm{d}\cos(\alpha_1))\sin(\omega t)\Big) = 0\,,$$

$$(\cos(\alpha_1) + f_\mathrm{d}\sin(\alpha_1))\,(1 - \cos(\omega t)) = (\sin(\alpha_1) - f_\mathrm{d}\cos(\alpha_1))\sin(\omega t)\,,$$

$$\tan\left(\frac{\omega t}{2}\right) = \frac{1-\cos(\omega t)}{\sin(\omega t)} = \frac{\sin(\alpha_1) - f_\mathrm{d}\cos(\alpha_1)}{\cos(\alpha_1) + f_\mathrm{d}\sin(\alpha_1)}\,,$$

$$\frac{\omega t}{2} = \arctan\left(\frac{\sin(\alpha_1) - f_\mathrm{d}\cos(\alpha_1)}{\cos(\alpha_1) + f_\mathrm{d}\sin(\alpha_1)}\right) = \arctan\left(\frac{f_\mathrm{s} - f_\mathrm{d}}{1 + f_\mathrm{d} f_\mathrm{s}}\right),$$

$$t_\mathrm{max} = \frac{2}{\omega}\arctan\left(\frac{f_\mathrm{s} - f_\mathrm{d}}{1 + f_\mathrm{d} f_\mathrm{s}}\right) \doteq 12.0\,\mathrm{s}\,.$$

In addition to the identity for the tangent of the half-angle, we also used the expressions for the sine of the angle α_1 as $f_\mathrm{s}/\sqrt{1+f_\mathrm{s}^2}$ and its cosine as $1/\sqrt{1+f_\mathrm{s}^2}$, which follow from the relation $\alpha_1 = \arctan(f_\mathrm{s})$.

Once we have expressed the time t_max, we can proceed to the final step – finding the position $x(t_\mathrm{max})$, which corresponds to the distance traveled by the point mass along the surface. Similarly to how we found $v(t)$ by integrating $a(t)$, we now express $x(t)$ by integrating $v(t)$ once more. We obtain

$$x(t) = \frac{g}{\omega^2}\Big(\sin(\alpha_1) - \sin(\alpha_1 - \omega t) - \omega t\cos(\alpha_1) + f_\mathrm{d}(\cos(\alpha_1 - \omega t) - \cos(\alpha_1) - \omega t\sin(\alpha_1))\Big)\,,$$

where the integration constant was determined from the condition $x(0) = 0$. Again, we use the sum formulas for sine and cosine and rewrite the relation into the form

$$x(t) = \frac{g}{\omega^2}\Big((f_\mathrm{d}\cos(\alpha_1) - \sin(\alpha_1))\,(\cos(\omega t) - 1) + (f_\mathrm{d}\sin(\alpha_1) + \cos(\alpha_1))\,(\sin(\omega t) - \omega t)\Big)\,,$$

which is again analogous to the expression for $v(t)$. Now it remains only to substitute for the time t_max and simplify the result into its final form. In the simplifications, we use the identities

$$\sin(2\arctan y) = \frac{2y}{1+y^2} \quad \text{and} \quad \cos(2\arctan y) = \frac{1-y^2}{1+y^2}\,.$$

We thus obtain

$$x(t_\mathrm{max}) = \frac{2g}{\omega^2}\left(\frac{f_\mathrm{s} - f_\mathrm{d}}{\sqrt{1+f_\mathrm{s}^2}} - \frac{1 + f_\mathrm{d} f_\mathrm{s}}{\sqrt{1+f_\mathrm{s}^2}}\arctan\left(\frac{f_\mathrm{s} - f_\mathrm{d}}{1 + f_\mathrm{d} f_\mathrm{s}}\right)\right) \doteq 27.7\,\mathrm{m}\,,$$

where we substituted the angular velocity as $\omega = \pi/180\,\mathrm{rad{\cdot}s^{-1}} \doteq 1.745\cdot 10^{-2}\,\mathrm{rad{\cdot}s^{-1}}$.

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Problem HB ... dramatic amplification

We place a conducting, uncharged sphere of radius $R = 7.5\,\mathrm{cm}$ *into a homogeneous electric field with intensity* $E = 333\,\mathrm{V{\cdot}m^{-1}}$. *Determine the maximum magnitude of the electric field in the region.*

Jarda noticed that when he sits next to the radio, it plays better.

When a conducting sphere is placed into an electric field, the charges rearrange themselves such that the surface of the sphere becomes an equipotential surface. To find the field, of course, we will use the following trick.

Consider inserting an electric dipole into the field instead. It has dipole moment $\mathbf{p}$ and is oriented along the z-axis, which is also the direction of the electric field. The potential due to a dipole is

$$\varphi_{\mathrm{dip}} = \frac{1}{4\pi\varepsilon}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon}\frac{pz}{r^3}\,,$$

where p is the magnitude of the dipole moment and $\mathbf{r}$ is the position vector measured from the origin at the center of the dipole. The total electric potential is

$$\varphi_{\mathrm{tot}} = \varphi_{\mathrm{dip}} - Ez = \left(\frac{1}{4\pi\varepsilon}\frac{p}{r^3} - E\right)z\,.$$

We can observe that for a certain distance R, for which the condition $p = 4\pi\varepsilon ER^3$ holds, the potential is zero independently of the value of z. Thus, around a dipole in a homogeneous electric field, an equipotential surface in the shape of a spherical shell exists. If we were to place a real conducting spherical shell on this surface, nothing would happen because the surface is equipotential. If we then remove the dipole inside the conducting sphere, the charges on the sphere rearrange themselves so that the surface is again equipotential. The situation outside the sphere, however, does not change by removing the dipole, because the field lines must still be perpendicular to the spherical surface. It can be shown that there is exactly one solution to such a problem–and we have found it. Placing a conducting sphere into a homogeneous electric field is (for the resulting field outside the sphere) equivalent to placing a dipole of suitable magnitude instead of the sphere at its center.

From the problem statement, the radius of the sphere is R, so the appropriate dipole moment is $p = 4\pi\varepsilon ER^3$. Such a dipole produces an electric field with intensity

$$\mathbf{E}_{\mathrm{dip}} = \frac{1}{4\pi\varepsilon}\left(3\frac{\mathbf{p}\cdot\mathbf{r}}{r^5}\mathbf{r} - \frac{\mathbf{p}}{r^3}\right) = ER^3\left(3\frac{\mathbf{z}\cdot\mathbf{r}}{zr^5}\mathbf{r} - \frac{\mathbf{z}}{zr^3}\right)\,,$$

where we have introduced the unit vector in the direction of the z-axis as $\mathbf{z}/z$, with $\mathbf{p} = p\cdot\mathbf{z}/z$. The total electric field intensity is

$$\mathbf{E}_{\mathrm{tot}} = \mathbf{E}_{\mathrm{dip}} + \mathbf{E} = \frac{ER^3}{zr^3}\left(3\frac{z^2}{r^2}\mathbf{r} + \left(\frac{r^3}{R^3} - 1\right)\mathbf{z}\right)\,.$$

We are looking for the maximum magnitude of the electric field, which is equivalent to finding the maximum of the square of this magnitude and then taking the square root at the end. We therefore compute

$$E_{\mathrm{tot}}^2 = \mathbf{E}_{\mathrm{tot}} \cdot \mathbf{E}_{\mathrm{tot}} = \left(\frac{ER^3}{zr^3}\right)^2 \left(9\frac{z^4}{r^4}r^2 + \left(\frac{r^3}{R^3} - 1\right)^2 z^2 + 6\frac{z^2}{r^2}\left(\frac{r^3}{R^3} - 1\right) z^2\right) =$$
$$= \left(ER^3\right)^2 \left(3\frac{z^2}{r^8} + \frac{1}{R^6} - \frac{2}{R^3 r^3} + \frac{1}{r^6} + 6\frac{z^2}{r^5 R^3}\right) .$$

Now consider a sphere of radius $r > R$. On each such sphere, the field magnitude is maximal at the points where z is maximal, i.e., for $z = r$. Substituting $z = r$, we obtain

$$E_{\mathrm{tot}}^2 = \left(ER^3\right)^2 \left(\frac{4}{z^6} + \frac{1}{R^6} + \frac{4}{z^3 R^3}\right) .$$

This function decreases rapidly with increasing z. It is therefore evident that the maximum magnitude of the electric field occurs just at the surface of the sphere, i.e., at the distance $z = R$. Substituting and taking the square root of the previous expression, we obtain the result

$$E_{\mathrm{max}} = 3E = 999\,\mathrm{V{\cdot}m^{-1}} ,$$

which does not depend on the radius of the sphere or on anything else.

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