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PVA EXPO PRAGUE

Solutions

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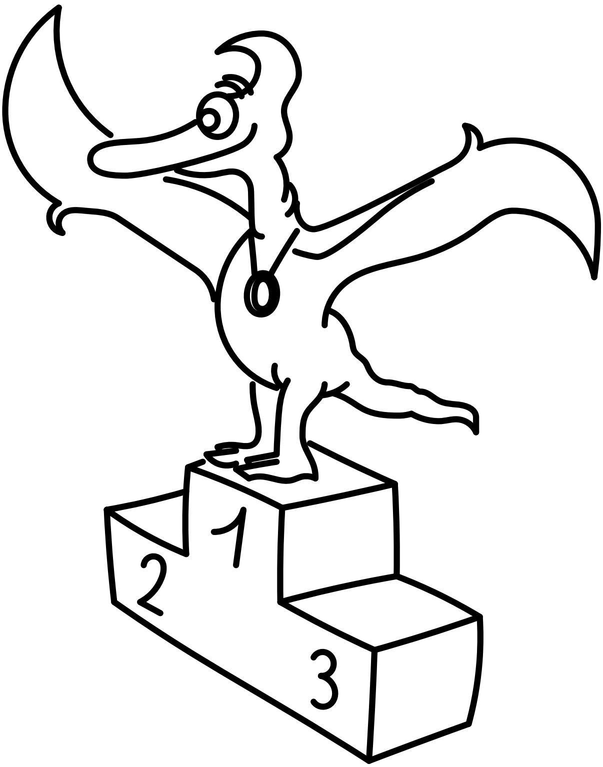
Solutions of problems



Problem AA ... quick pasta

Honza is cooking pasta for dinner in his dorm room. He has at his disposal a cooker of power $P_1 = 1\,200\,\mathrm{W}$ and a kettle of power $P_2 = 2\,200\,\mathrm{W}$. What is the shortest time in which he is able to heat half a liter of water from $20\,^\circ\mathrm{C}$ to $100\,^\circ\mathrm{C}$? *Honza was in a hurry to get dinner ready.*

First, let us determine how long it would take a theoretical appliance of power $P = P_1 + P_2$

$$P = \frac{Q}{t} \quad \Rightarrow \quad t = \frac{Q}{P} = \frac{Q}{P_1 + P_2}\,,$$

where Q is the heat required for heating the water and t is the time in which the water is able to be heated by the theoretical appliance. We can consider that we only have this one theoretical appliance, because if we divided the water in such a way that it would start to boil faster in one appliance than in the other one, we wouldn't be using the power at our disposal wisely.[1]

To calculate the required heat, we use the following equation

$$Q = m c_{\mathrm{water}} \Delta T\,,$$

where $m = V \rho_{\mathrm{water}}$ is the mass of half a liter of water, ΔT is the temperature difference between the final and initial state of water, so $\Delta T = 100\,^\circ\mathrm{C} - 20\,^\circ\mathrm{C} = 80\,^\circ\mathrm{C}$ and c_{water} is the specific heat capacity of water.

Combining the two previous equations, we obtain

$$t = \frac{V \rho_{\mathrm{water}} c_{\mathrm{water}} \Delta T}{P_1 + P_2} \doteq 49\,\mathrm{s}\,.$$

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Problem AB ... more panic on the escalators

We are riding up an escalator moving at speed $u = 0.75\,\mathrm{m\cdot s^{-1}}$. When we are at two-thirds of its total length s, we suddenly realize that we need to get back down as fast as possible. What is the minimum speed we would need to run at, for it to be faster to turn around and run down the escalator, rather than to run up to the top of and then run down the escalator moving in the opposite direction? Ignore the people on the escalators and the run-over time at the top. *Karel was still thinking about what to do.*

If we decide at two-thirds of the escalator to return down, we have two options: either run the same distance against the motion of the escalator or run four-thirds of the distance s in the direction of the moving stairs (one-third up and the whole staircase back down).

[1] We can also imagine that once the water starts to boil in one of the appliances, we pour some of it in the other one. This way, we are always using the full power, which is when the time will be the shortest.

When running against the stairs, our velocity relative to the surroundings is $w_1 = v - u$. When running in the direction of the stairs, our velocity is $w_2 = v + u$. For it to be faster to turn around and run down directly, the following inequality must hold

$$t_1 < t_2 \,,$$
$$\frac{\frac{2}{3}s}{w_1} < \frac{\frac{4}{3}s}{w_2} \,,$$
$$\frac{1}{v-u} < \frac{2}{v+u} \,,$$
$$v + u < 2v - 2u \,,$$
$$v > 3u \doteq 2.3\,\mathrm{m{\cdot}s^{-1}} \,.$$

Therefore, to get back down as quickly as possible, it is worth turning around if we run faster than three times the speed of the escalator, that is $2.3\,\mathrm{m{\cdot}s^{-1}}$ (or $8.1\,\mathrm{km{\cdot}h^{-1}}$).

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Problem AC ... gasoline magic

Three FYKOS members went on a trip to Vienna. While estimating the price of gasoline, they had a disagreement. The physicist's estimation was 1 500 Kč, while the computer scientist's was 2 665 Kč. After an intense discussion, it became clear that the computer scientist did the following calculation: he took the distance to Vienna in kilometers, divided it by the consumption of the car in liters per 100 km and multiplied it by the price of gasoline, 30 Kč per liter, conjuring up utter nonsense.

The physicist, completely baffled as to how he could come up with a lower estimate than the computer scientist, even when assuming the price of gasoline was 40 Kč per liter, quickly found the error. Although they disagreed on the calculation method, they both coincidentally estimated the same consumption for the car. What is the car's fuel consumption in liters per 100 km? *Radek was interrogated regarding his calculations.*

Let us denote Σ the computer scientist's estimate, σ the physicist's estimate, s the distance to Vienna and V the car's consumption per 100 km. For Σ and σ the following holds

$$\Sigma = 30\frac{s}{V} \,,$$
$$\sigma = \frac{40}{100}sV \,.$$

Let us now express the distance to Vienna s from both equations

$$s = \frac{1}{30}\Sigma V \,,$$
$$s = \frac{100}{40}\frac{\sigma}{V} \,.$$

The resulting equations can now be put into an equation, expressing V as

$$\frac{1}{30}\Sigma V = \frac{100}{40}\frac{\sigma}{V}\,,$$
$$V = \sqrt{75\frac{\sigma}{\Sigma}}\,.$$

Now by simple substitution and rounding we get

$$V \doteq 6.5\,\mathrm{l\cdot(100\,km)^{-1}}\,.$$

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Problem AD ... slow down, slow down!

We want to stop a car on a flat surface at a distance of $d = 50.0\,\mathrm{m}$. We know that the car's wheels do not slip only when its acceleration is less than $a = 0.780g$. What is the highest velocity the car can travel at to stop in time?

Karel thought that a classic would be a good place to start.

Firstly, if the car's wheels were slipping, the braking effect would be diminished, meaning the car would brake with the maximum specified acceleration a. The maximum deceleration occurs when the car maintains this acceleration throughout. Now, we want the car to stop within a distance shorter than d. Therefore, the following inequality must hold

$$d > \frac{1}{2}at^2\,.$$

The braking time is simply

$$t < \sqrt{\frac{2d}{a}}\,.$$

The maximum velocity the car can have before braking begins is

$$v = at = \sqrt{2da} \doteq 27.7\,\mathrm{m\cdot s^{-1}} \doteq 99.6\,\mathrm{km\cdot h^{-1}}\,.$$

Therefore, if the car starts decelerating optimally 50 meters from the obstacle and can decelerate with an acceleration of $0.780g$, it will be able to stop if its speed is below $99.6\,\mathrm{km\cdot h^{-1}}$.

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Problem AE ... deeply they wave

Pepa sailed his private yacht during the holidays to watch the waves. He noticed that far from the coast – in the so-called deep water *– the effect of water depth is not significant. It can therefore be assumed that the angular frequency of the waves ω depends solely on the gravitational acceleration g and the wavelength λ. Deduce what this dependence should look like, i.e., find the real numbers α and β such that $\omega = Cg^{\alpha}\lambda^{\beta}$, where C represents some dimensionless constant. The relation obtained is equivalent to the so-called dispersion relation for waves in deep water.*

Pepa was bored in Croatia.

The dispersion relation is the relationship between the wavelength λ and the angular frequency ω. According to the assignment, we do not assume that ω depends on the depth of the water but only on the specified quantities, i.e., the wavelength λ and the gravitational acceleration g.

The principle of dimensional analysis is that on both sides of the equation for ω, the terms must have the same unit, so in this case, $\mathrm{Hz} = \mathrm{s}^{-1}$.

Suppose the relationship is in the form

$$\omega = C g^{\alpha} \lambda^{\beta} ,$$

where C is some dimensionless constant that we cannot determine by dimensional analysis. For this expression to be meaningful, we must find the numbers α and β so that the units on the right-hand and left-hand sides of the equation are the same. Symbolically, for the units (without the constant C), we write the following

$$\mathrm{m}^0 \cdot \mathrm{s}^{-1} = \mathrm{m}^{\alpha} \cdot \mathrm{s}^{-2\alpha} \cdot \mathrm{m}^{\beta} .$$

Here, we symbolically write $\mathrm{m}^0 = 1$ on the right-hand side because meters do not appear in the unit for ω. By comparing the exponents of seconds, we obtain the algebraic equation $-1 = -2\alpha$, and for meters, $0 = \alpha + \beta$. Solving this simple system, we find $\alpha = 1/2$ and $\beta = -1/2$. Thus, (without the constant), it must hold that

$$\omega \propto \sqrt{\frac{g}{\lambda}} .$$

For the record, if we were to solve the wave equation using the appropriate simplifications within Airy's linearized theory, we would get the exact relationship (including the constant) for waves on water of depth $h \gg \lambda/2$

$$\omega = \sqrt{\frac{2\pi g}{\lambda}} .$$

We can see that the relationship is nonlinear, which leads to the dependence of the wave velocity factor on the wavelength (the so-called dispersion). More precisely, the following relation holds

$$\omega = \sqrt{\frac{2\pi g}{\lambda} \tanh \frac{2\pi h}{\lambda}} ,$$

where in deep water, we neglect the hyperbolic tangent as $\tanh\left(2\pi h/\lambda\right) \approx 1$. Thus, we see that waves of different lengths (at least within the approximation) move at different speeds, whether they originate from gravity, surface tension, or other possible factors.

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Problem AF ... calculating friction

We place a block of mass $M = 5.5\,\mathrm{kg}$ on the table and connect it with a rope and pulley to a block of mass $m = 1.1\,\mathrm{kg}$ hanging next to the table. The coefficient of static friction between the block on the table and the table is $f_\mathrm{s} = 0.47$, and the coefficient of dynamic friction is $f_\mathrm{d} = 0.27$. The blocks are initially at rest. How large will the frictional force between the block and the table be? *Lego thinks there has never been a catch like this before.*

The key idea for solving this problem is that the equation $F_\mathrm{t} = fF_\mathrm{n}$ describes the maximal possible frictional force, which works against the direction of motion. Therefore, when everything is stationary, the force is just large enough for everything to stay stationary.

In our case, the maximal possible frictional force is $F_\text{t-max} = f_\mathrm{s}Mg \doteq 25\,\mathrm{N}$. However, the block is pulled to the side with force $F_\mathrm{k} = mg \doteq 11\,\mathrm{N} < F_\text{t-max}$, which makes the frictional force $F_\mathrm{t} = F_\mathrm{k}$. If the force were any larger, it would put the block in motion in the opposite direction, which is not how friction works because it would immediately have to reverse its direction.

The block remains stationary because $F_\mathrm{k} < F_\text{t-max}$ and friction is able to hold it in place. The answer to the problem is that the frictional force is $11\,\mathrm{N}$.

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Problem AG ... carbon emissions zero

Fykosaurus was captivated by the majestic cooling towers as he flew around the Temelín nuclear power plant. From all the towers combined, water vapor was escaping at a volumetric flow rate of Q. A moment later, Fykosaurus flew past the coal-fired power plant, where the cooling towers were also emitting steam at a volume flow of Q. Fykosaurus wondered: how many times more mass of coal must be burned in the coal power plant compared to the mass of uranium-235 that undergoes fission in the nuclear power plant to evaporate the same amount of water? The average usable energy released from the fission of a single uranium-235 nucleus is $200\,\mathrm{MeV}$. The molar mass of the uranium-235 isotope is $235\,\mathrm{g{\cdot}mol^{-1}}$. The maximum calorific value of black coal is $30\,\mathrm{MJ{\cdot}kg^{-1}}$. Assume that all thermal energy generated in the power plant is used to evaporate water. The water is heated from a temperature of $20\,\mathrm{^\circ C}$ in both power plants. Neglect the change in specific heat capacity of water with increasing temperature.

This problem is brought to you by the CEZ Group.

Jindra would rather carry a gram of uranium to the power plant than a ton of coal.

The energy released per unit mass of uranium-235 isotopes is

$$H_\mathrm{U} = \frac{E_\mathrm{U}}{m_\mathrm{U}} = \frac{N_A E_0}{M_{235}},$$

where $E_0 = 200\,\mathrm{MeV} = 3.20 \cdot 10^{-11}\,\mathrm{J}$ is the average extractable energy released during the decay of a uranium-235 nucleus, M_{235} is the molar mass of the uranium-235 isotope, and $N_A = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$ is the Avogadro's constant, which can be found in the list of constants.

From the problem statement, we know that the energy released when combusting a unit weight of good quality coal is

$$H_\mathrm{c} = \frac{E_\mathrm{c}}{m_\mathrm{c}} = 30\,\mathrm{MJ{\cdot}kg^{-1}}.$$

To heat up a given amount of water, $E_\mathrm{U} = E_\mathrm{c}$ must hold

$$\frac{N_A E_0}{M_{235}} m_\mathrm{U} = H_\mathrm{c} m_\mathrm{c} ,$$

$$\frac{m_\mathrm{c}}{m_\mathrm{U}} = \frac{N_A E_0}{M_{235} H_\mathrm{c}} \doteq 2.7 \cdot 10^6 .$$

The mass of coal required for the vaporization of water is $2.7 \cdot 10^6$ times greater than the mass of the uranium-235 isotope.

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Problem AH ... pressure in an enclosed tube

Consider an enclosed U-tube with a constant cross-section S. Its ends are at different heights, their difference being Δh. The tube contains only water and an air bubble with a volume of V, which is in the “shorter end” of the tube in the beginning. We measure the pressure on the bottom of the tube. Then we rotate the tube (without opening it) and let the air bubble move to the longer end. Afterwards we return the U-tube to its original position and once again measure the pressure at the bottom of the tube. How has the pressure changed? You can assume that the air stays at a constant temperature, that the bubble always settles down in the vertical part of the tube, and that the walls of the U-tube are perfectly rigid.

Lego found it on YouTube.

Considering that the water does not change its volume much and the walls of the tube are perfectly solid, meaning that the tube does not change its volume, the air will not change its volume either, because its volume is given by the difference between the volume of the tube and the water in it. At the same time, we assume that the air temperature does not change and considering that the tube is closed, the amount of air in the tube cannot change either. From the equation of state of the air in the tube

$$pV = nRT ,$$

we see that we already know that everything except the pressure are constants. Therefore, the pressure must also remain constant at all times; otherwise, the equation of state would not always be satisfied.

Let us denote the pressure in the air bubble by p_0. Then the pressure at the bottom of the U-tube will be p_0, to which we have to add the hydrostatic pressure from the column of water between the bottom of the bubble and the bottom of the U-tube, which is $p_0 + h_1 \rho g$. Since we are only interested in the difference between this pressure at two different positions of the bubble, p_0 is canceled out and only the difference of hydrostatic pressures remains.

Since the volume of the bubble does not change and the cross-section of the tube is constant, the height of the bubble itself will be the same in both cases. The pressure under the longer end is therefore $p_2 = p_0 + h_2 \rho g$, where the hydrostatic pressure is caused by the column of water h_2 in the longer end. It is now higher by Δh than when the bubble was in the shorter end, so $h_2 - h_1 = \Delta h$.

Thus the pressure will be greater in the second case (when the bubble is in the longer end of the tube) and will be greater by exactly

$$\Delta p = p_2 - p_1 = p_0 + h_2 \rho g - (p_0 + h_1 \rho g) = \Delta h \rho g\,.$$

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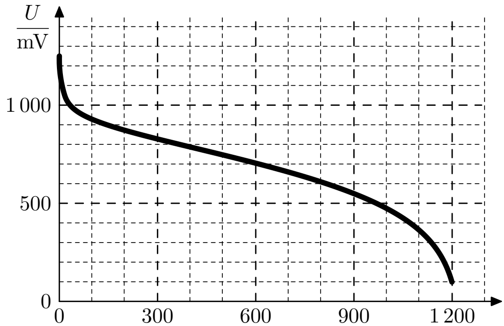
Problem BA ... fuel cell

A fuel cell is a device that produces electricity from fuel through electrochemical reactions. In the current effort to switch to renewable energy sources, we may hear about hydrogen fuel cells, which would use hydrogen H_2 *reacting with atmospheric oxygen to produce water and electricity. Such a device acts as a source of electric current and voltage, where both quantities are linked by a polarization curve. Since fuel cells can vary in size, the current is expressed as the current density* j*, which represents the current per unit area of the cell. Using the graph, determine the maximum surface power density that the given fuel cell can provide, with a tolerance of* $\pm 20\,\mathrm{mW{\cdot}cm^{-2}}$.



Jarda promotes electrochemistry across all competitions.

We can express the surface power density (or surface power for short) of a power source as $P = Uj$, where U is the voltage across the cell. In the graph, we plotted the dependence of the voltage on the current density, so we have both required quantities. Thus, we just need to multiply them for several points on the graph and submit the highest value as the solution to the problem.

At the edges of the graph, the surface power is zero because either the voltage or the current density is zero. We can suspect that the curve will only have one maximum. Let's successively substitute for chosen points on the graph. For $300\,\mathrm{mA{\cdot}cm^{-2}}$, the voltage is about $830\,\mathrm{mV}$, which corresponds to about $250\,\mathrm{mW{\cdot}cm^{-2}}$. For $600\,\mathrm{mA{\cdot}cm^{-2}}$ the voltage is about $700\,\mathrm{mV}$, which corresponds to about $420\,\mathrm{mW{\cdot}cm^{-2}}$, which is significantly more. For $900\,\mathrm{mA{\cdot}cm^{-2}}$, a power output is about $500\,\mathrm{mW{\cdot}cm^{-2}}$. So, the maximum will be somewhere around this value of current density. At $1\,100\,\mathrm{mA{\cdot}cm^{-2}}$ the power is again significantly lower, so we can expect the maximum to be between $700\,\mathrm{mA{\cdot}cm^{-2}}$ and $1\,000\,\mathrm{mA{\cdot}cm^{-2}}$. Let's try $750\,\mathrm{mA{\cdot}cm^{-2}}$, where we estimate the voltage to be $640\,\mathrm{mV}$, which leads to about $480\,\mathrm{mW{\cdot}cm^{-2}}$. For $800\,\mathrm{mA{\cdot}cm^{-2}}$, this is then about the same value, at $850\,\mathrm{mA{\cdot}cm^{-2}}$ we estimate the voltage to be $580\,\mathrm{mW}$, which leads to $490\,\mathrm{mW{\cdot}cm^{-2}}$.

The maximal calculated surface power was $500\,\mathrm{mW{\cdot}cm^{-2}}$ at $900\,\mathrm{mA{\cdot}cm^{-2}}$. The analytically calculated maximum for our function is $495\,\mathrm{mW{\cdot}cm^{-2}}$ for $883\,\mathrm{mA{\cdot}cm^{-2}}$. We can see that in just a few steps, we have gotten very close to the correct result, which is $(495 \pm 20)\,\mathrm{mW{\cdot}cm^{-2}}$ within the tolerance interval.

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Table 1: Values taken from the graph.

$\frac{j}{\mathrm{mA\cdot cm^{-2}}}$	$\frac{U}{\mathrm{mV}}$	$\frac{P}{\mathrm{mW\cdot cm^{-2}}}$
300	830	250
600	700	420
900	550	500
1 000	480	480
1 100	370	410
750	640	480
800	600	480
850	580	490

Problem BB ... microscopic expansion

The movement of the tip in the scanning tunneling microscope can be controlled with an accuracy of $0.10\,\mathrm{Å}$ *(where* Å *indicates the unit of angstrom;* $1\,\mathrm{Å} = 10^{-10}\,\mathrm{m}$*) in the direction perpendicular to the sample surface. What could be the maximal value of the coefficient of thermal expansion for a tip holder whose length is* $300\,\mathrm{\mu m}$*, so that when the holder's temperature changes by* $0.10\,\mathrm{K}$*, the change in the position of the tip is not greater than the above-mentioned accuracy? This holder, like the tip itself, stands perpendicular to the surface. The Nobel Prize for the invention of scanning tunneling microscope was awarded almost 40 years ago.*

For a small range of temperature change ΔT, the change in length can be expressed using a linear relationship

$$\Delta l = l_0 \alpha \Delta T\,,$$

where l_0 is the original length and α is the temperature expansion coefficient that we are trying to find. Everything happens on a straight line, perpendicular to the surface of the sample. It is therefore possible to express α and substitute the quantities from the problem statement

$$\alpha = \frac{\Delta l}{l_0 \Delta T} = \frac{0.01\,\mathrm{nm}}{300\,\mathrm{\mu m}\cdot 0.1\,\mathrm{K}} \doteq 3.3\cdot 10^{-7}\,\mathrm{K^{-1}}.$$

The resulting value of α is low compared to commonly available materials. The holder must be made from a suitable material and the temperature needs to be stabilized during the measurements. Alternatively, the change in length can also be corrected electronically. An atomic scale resolution can be achieved in this microscope, which allows us to study the electronic structure of the individual atoms on the surface.

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Problem BC ... exploring extraterrestrial puddles

A probe in the shape of a rectangular cuboid with a mass of m *lands on a completely unknown planet. It lands in an unknown fluid and remains floating in it so that it is submerged to a depth of* h *and its base* S *is horizontal. The probe manages to measure that the pressure at its bottom base is* p *and that the atmospheric pressure is negligible. It also measures that the* *density of the fluid in which it is submerged increases linearly with depth (at least to the depth to which the probe is submerged). What is the planet's gravitational acceleration?*

Lego simply wanted a pressure-based Archimedes problem.

The buoyancy force is just the sum of all the compressive forces acting on the body. The forces exerted by the fluid pushing against the side walls of the probe cancel each other out. Thus, we will only talk about the difference in pressure on the upper and lower bases. Since the assignment states that the atmospheric pressure is negligible, there is no pressure acting on the top base of the probe, leaving us with just the bottom base. The bottom base has the area S and the pressure acting on the surface is equal to p, so the force is equal to $F_p = Sp$, which is, therefore also the magnitude of the resulting buoyancy force. The gravitational force acting on the probe must be of the same magnitude, and since we know the mass of the probe, we simply determine the gravitational acceleration as $g_n = F_g/m = Sp/m$.

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Problem BD ... stretching a pulley

In a room of height $h = 300\,\mathrm{cm}$, a spring of rest length $l_0 = 20\,\mathrm{cm}$ and a stiffness of $k = 50\,\mathrm{N{\cdot}m^{-1}}$ hangs from the ceiling. At its end is suspended a massless pulley of radius $r = 10\,\mathrm{cm}$, attached by its center. Over the pulley we throw an inelastic massless rope of length $L = 400\,\mathrm{cm}$ and secure both ends to the ground (so that it hangs vertically). What is the force applied on the rope?

Lego wanted to steal this problem statement, but then no one else used it.

First, we need to calculate the elongation of the spring. There is πr of the rope wound on the pulley and the rest is divided into two identical hlaves, which run vertically downwards. Thus, the centre of the pulley is located at the height

$$h_1 = \frac{L - \pi r}{2}$$

above the ground. The length of the spring is therefore $h - h_1$, and hence the extension of the spring is $h - h_1 - l_0$.

From this we calculate the force exerted by the spring on the pulley upwards as

$$F_k = k\left(h - l_0 - \frac{L - \pi r}{2}\right) .$$

This force is distributed evenly on both sides of the rope, therefore the rope will be strained by the force

$$F_T = \frac{k}{2}\left(h - l_0 - \frac{L - \pi r}{2}\right) \doteq 24\,\mathrm{N} .$$

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Problem BE ... conductance quantum

Perhaps you have heard of Planck's units, where physical units are defined using only the universal constants c, G, k_{B} and $\hbar$. If we include the elementary charge e, we could further extend the list of possible units. Interestingly, some units defined in this way appear in various physical theorems. One such unit is the "Planck's" conductance (not to be confused with conductivity, which is conductance per unit length). Your task is to find the value of Planck's conductance in SI units, which appears, for example, in the quantum tunneling theorem.

Jarda wanted to devise a problem for dimensional analysis and just encountered this.

Let's use the so-called dimensional analysis to solve this problem. In this process, we look at the units of the quantities available to us (c, G, k_{B}, $\hbar$ and e) and combine them in such a way that the resulting unit is the same as the unit of the quantity we are looking for.

In SI units, the units of the given quantities are the following

$$\begin{aligned}[c] &= \mathrm{m{\cdot}s^{-1}}\,,\\ [G] &= \mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}\,,\\ [k_{\mathrm{B}}] &= \mathrm{m^2{\cdot}kg{\cdot}s^{-2}{\cdot}K^{-1}}\,,\\ [\hbar] &= \mathrm{m^2{\cdot}kg{\cdot}s^{-1}}\,,\\ [e] &= \mathrm{C}\end{aligned}$$

and the unit of the conductance is

$$[\sigma] = \Omega^{-1} = \mathrm{kg^{-1}{\cdot}m^{-2}{\cdot}s^{1}{\cdot}C^{2}}\,.$$

We see that the unit of conductance does not involve the kelvin unit, so there is no need to use the Boltzmann constant. Also, there is a square of the coulomb unit, so the sought relationship will certainly involve e^2. Now, we write three equations with three unknowns to deduce what exponents must be applied to the speed of light, the gravitational constant, and the reduced Planck constant, so that the combined product of its units gives the rest of the conductance unit. If we notice that this "rest" is equal to the reciprocal of the reduced Planck constant's unit, we can simply write

$$\sigma_{\mathrm{kvant}} = \frac{e^2}{\hbar}\,, \quad [\sigma_{\mathrm{kvant}}] = \mathrm{C^2} \cdot \left(\mathrm{m^2{\cdot}kg{\cdot}s^{-1}}\right)^{-1} = \mathrm{kg^{-1}{\cdot}m^{-2}{\cdot}s^{1}{\cdot}C^{2}}\,.$$

By inserting the numerical values, we get

$$\sigma_{\mathrm{kvant}} = \frac{e^2}{\hbar} \doteq 2.43 \cdot 10^{-4}\,\mathrm{kg^{-1}{\cdot}m^{-2}{\cdot}s^{1}{\cdot}C^{2}} = 2.43 \cdot 10^{-4}\,\Omega^{-1}\,.$$

Fun fact: this value corresponds to a macroscopic resistor with a resistance of about $4\,\mathrm{k\Omega}$.

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Problem BF ... an attempt at (not)shooting oneself

How much time do you have to dodge a bullet you fire on a planet or moon with no atmosphere? You are shooting in such a way that the bullet remains at a constant (and negligible) height above the surface of a perfectly spherical body of radius R and mass M on which you are standing. *Karel had some ideas.*

The gravitational force attracting the bullet to the centre of the planet is

$$F_G = G\frac{Mm}{R^2}\,,$$

where G is Newton's gravitational constant and m is the mass of the bullet. No other force is expected to act on the bullet. Thus, if we want the bullet to fly along a circle of radius R, we must shoot it at a velocity v for which the corresponding centripetal force $F_d = mv^2/R$ is equal to the gravitational force. Let us put these two forces into equation

$$F_G = F_d\,,$$
$$G\frac{Mm}{R^2} = m\frac{v^2}{R}\,,$$
$$\sqrt{G\frac{M}{R}} = v\,.$$

The distance the bullet travels before hitting us is $s = 2\pi R$, so the time we have for dodging is

$$t = \frac{s}{v} = \frac{2\pi R}{\sqrt{G\frac{M}{R}}} = 2\pi\sqrt{\frac{R^3}{GM}}\,.$$

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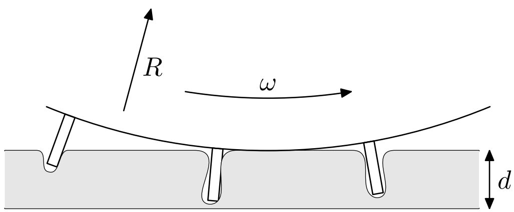
Problem BG ... peristaltic pump

Consider a thin tube made of a very flexible material with inner diameter d. This tube passes through a peristaltic pump which contains a rotating ring that has N rods evenly spaced around its circumference. By rotating the whole ring, the rods compress the tube and move the water forward (see figure). With what frequency f must the ring of radius R rotate to generate a volume flow Q through the tube? Consider $d \ll R$ and neglect the compressed volume of the tube. *These pumps deliver water to the electrolyzers in Jarda's lab.*



The water in the tube moves at the same speed as the circumferential velocity of the circle, which is $v = 2\pi fR$.

The volumetric flow rate is then

$$Q = Sv = \frac{\pi d^2}{4} 2\pi R f = \frac{\pi^2}{2} R d^2 f\,.$$

From here, we can express the frequency f as

$$f = \frac{2Q}{\pi^2 R d^2}\,.$$

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Problem BH ... pineapple on pizza

Pineapple slices were thrown with the same initial velocity at angles $\alpha = 38^\circ$, $\beta = 45^\circ$ and $\gamma = 52^\circ$ relative to the horizontal direction so that all fell on a pizza. The pizza has radius of 16 cm and its center is 3.0 m from the point of ejection. Determine the interval of initial velocities for which all three slices land on the pizza.

David is already dreaming about Hawaii.

The pineapple slices are thrown under angles $\alpha = 38^\circ$, $\beta = 45^\circ$ and $\gamma = 52^\circ$, so it is a projectile motion. For an oblique throw, the distance of impact from the throwing point can be derived as

$$D = \frac{v_0^2 \sin(2\xi)}{g}\,,$$

for an arbitrary angle ξ. Let us denote the distance of the pizza as s and the radius of the pizza as r, then the interval of the allowed distances D into which the slices can land is equal to $[s - r;\, s + r]$. We can also notice that $\sin(2\alpha) = \sin(2\gamma)$, thus we just need to calculate the allowed initial velocities

$$v_0 = \sqrt{\frac{Dg}{\sin(2\xi)}}$$

for only two angles.

After substituting the extremes of the allowed values of D for the angles α and γ, the interval of the initial velocity v_α is

$$v_\alpha \in [5.4;\, 5.7]\,\mathrm{m{\cdot}s^{-1}}\,,$$

and for the angle β the interval of the initial velocity v_β

$$v_\beta \in [5.3;\, 5.6]\,\mathrm{m{\cdot}s^{-1}}\,.$$

Our resulting velocity v will therefore lie in the intersection of the intervals v_α and v_β, i.e.

$$v \in [5.4;\, 5.6]\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem CA ... hard impact

We're watching two identical cars driving in the same direction on a highway. One is travelling at $v_1 = 80\,\mathrm{km{\cdot}h^{-1}}$, the other at $v_2 = 130\,\mathrm{km{\cdot}h^{-1}}$. When the faster car is on the same level as the slower car while overtaking, both drivers notice an obstacle in front of them. They both start braking with maximum braking force, which is constant and equal for both cars. The slower car manages to stop right before the obstacle. What is the speed of the other car when it hits the obstacle? *Karel watched videos.*

The traveled distance of the first car that stopped just before the obstacle was s. It is also the distance that the second car has for braking. Let us denote the mass of the car by m. Then, the braking force $F = ma$ performs work $W = Fs$. This amount of work is the same for both cars. The slower car loses all of its kinetic energy

$$\frac{1}{2}mv_1^2 = Fs\,.$$

Let's calculate the final velocity of the second car v_f from the change of its kinetic energy

$$\frac{1}{2}mv_\mathrm{f}^2 = \frac{1}{2}mv_2^2 - Fs = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad \Rightarrow \quad v_\mathrm{f} = \sqrt{v_2^2 - v_1^2} \doteq 102\,\mathrm{km{\cdot}h^{-1}}\,.$$

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Problem CB ... a barking dog never bites

Jarda's dogs are ill-mannered and often bark at passing cyclists. One such cyclist, riding towards them, heard their barking with a period of $T_1 = 1.4\,\mathrm{s}$. After riding past them, he continued on his way and, luckily, heard the barking again, but this time with a period of $T_2 = 1.5\,\mathrm{s}$. Determine the cyclist's speed, assuming he was moving in a straight line at a constant speed and that the dogs' barking occurs with a non-changing period.
Jarda's online classes were disturbed by barking dogs.

This problem is an example of a classic Doppler effect. Let $c = 343\,\mathrm{m{\cdot}s^{-1}}$ denote the speed of sound in air, v the speed of the cyclist, and T the barking period. The barking frequency is therefore $f = 1/T$. As the cyclist approaches, the frequency of barking he hears is equal to

$$f_1 = f\left(1 + \frac{v}{c}\right)\,,$$

on the contrary, when moving away, it is

$$f_2 = f\left(1 - \frac{v}{c}\right)\,.$$

By dividing one equation by the other and further rearranging the terms, we get

$$v = c\frac{f_1 - f_2}{f_1 + f_2} = c\frac{T_2 - T_1}{T_1 + T_2}\,,$$

so the velocity of cyclist is $v \doteq 12\,\mathrm{m{\cdot}s^{-1}}$.

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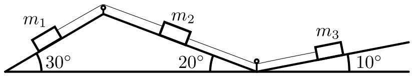
Problem CC ... hill climb racing



Let's have three inclined planes. The first one rises with slope $\alpha_1 = 30°$, and after a sharp peak it flips into another one that falls with slope $\alpha_2 = 20°$. After a sharp minimum, this flips into a plane rising with slope $\alpha_3 = 10°$. On the respective planes, there are blocks with masses $m_1 = 20\,\mathrm{kg}, m_2 = 10\,\mathrm{kg}, m_3 = 15\,\mathrm{kg}$, which are connected by weightless ropes and pulleys (the ropes always find themselves parallel to the hill, see figure). With what magnitude of acceleration will a block of mass m_3 be moving? Do not consider friction. *Lego completed a ridge trail.*

As long as we neglect friction, it is sufficient to consider only the forces parallel to each inclined plane for each block. We will denote the tension in the rope between m_1 and m_2 as T_{12}, and between m_2 and m_3 as T_{23}. Then, the equation of motion for the first block has the form

$$m_1 g \sin\alpha_1 - T_{12} = m_1 a_1\,,$$

where we consider the downward direction as positive.

For the second block, we will define the upward direction as positive, therefore, a_2 is positive exactly when a_1 is also positive. Then, its equation of motion is

$$T_{12} - T_{23} - m_2 g \sin\alpha_2 = m_2 a_2\,,$$

and finally, for the third block, we will again define the downward direction as positive

$$m_3 g \sin\alpha_3 + T_{23} = m_3 a_3\,.$$

Now, we have 3 equations and 5 unknowns: $a_1, a_2, a_3, T_{12}, T_{23}$. However, we have yet to use a fundamental property of the rope – its inextensibility. This constraint implies that the accelerations of the connected masses must be equal, meaning $a_1 = a_2$ (Equation 4). Additionally, if the rope remains taut at all times, as we have assumed, then $a_2 = a_3$. Substituting these relationships between accelerations into our system of equations simplifies it. Let's proceed with solving the system and then analyze the result

$$\begin{aligned} m_1 g \sin\alpha_1 - T_{12} &= m_1 a_3 \\ T_{12} - T_{23} - m_2 g \sin\alpha_2 &= m_2 a_3 \\ m_3 g \sin\alpha_3 + T_{23} &= m_3 a_3\,. \end{aligned}$$

When we sum all the equations, we eliminate the tensions and get a very intuitive result

$$m_1 g \sin\alpha_1 - m_2 g \sin\alpha_2 + m_3 g \sin\alpha_3 = (m_1 + m_2 + m_3) a_3$$

$$\Rightarrow \quad a_3 = g\frac{m_1 \sin\alpha_1 - m_2 \sin\alpha_2 + m_3 \sin\alpha_3}{m_1 + m_2 + m_3}\,.$$

One way to check if the assumptions we used are correct would be to substitute a_3 back into the third equation of motion and calculate the value of T_{23}. If it is positive, this confirms that the rope remains taut and applies a pulling force on the blocks. However, if T_{23} is negative, our equations suggest that the rope pushes the blocks, which is impossible. This means our assumption that the rope remains taut must be incorrect.

Thankfully, it is unnecessary to substitute and do some complicated mathematical operations. Let us express T_{23} first

$$T_{23} = m_3(a_3 - g\sin\alpha_3)\,.$$

It will be positive if $a_3 > g\sin\alpha_3$, and negative otherwise. This makes sense when we realize that $g\sin\alpha_3$ is the acceleration with which the block would accelerate downhill if it were not being pulled or slowed down by anything. Since the rope pulls the block, we find that a_3 is greater than this value, confirming our assumption. However, if $a_3 < g\sin\alpha_3$, the result would be unphysical, as the rope cannot slow down the third block. In that case, the correct acceleration would simply be $g\sin\alpha_3$. Now, we only need to plug into all of the equations

$$\begin{aligned} a_3^{(\text{tense})} &= g\frac{m_1\sin\alpha_1 - m_2\sin\alpha_2 + m_3\sin\alpha_3}{m_1+m_2+m_3} \doteq 2.0\,\mathrm{m{\cdot}s^{-2}}\,,\\ a_3^{(\text{loose})} &= g\sin\alpha_3 \doteq 1.7\,\mathrm{m{\cdot}s^{-2}}\,. \end{aligned}$$

And thus

$$a_3^{(\text{tense})} > a_3^{(\text{loose})} \quad\Rightarrow\quad a_3 = a_3^{(\text{tense})} \doteq 2.0\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem CD ... clear ice

How big of an area can we de-ice using a $30\,\mathrm{l}$ *butane bottle with a pressure of* $0.20\,\mathrm{MPa}$*? The butane in the bottle has a temperature of* $20\,\mathrm{°C}$ *and a molar mass of* $58\,\mathrm{g{\cdot}mol^{-1}}$*. The burning of butane has a specific energy of* $46\,\mathrm{MJ\cdot kg^{-1}}$*. The frost is formed by a layer of ice with a density* $\rho_{led} = 916.8\,\mathrm{kg{\cdot}m^{-3}}$*, specific heat capacity* $c = 2\,090\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ *and thickness* $1.0\,\mathrm{cm}$*. Because it is cold, the frost (and the surrounding air) has a temperature of* $-5.0\,\mathrm{°C}$*. Consider perfect heat transfer and butane in the bottle as the ideal gas.*

David saw the sidewalks in Prague being de-iced.

First, we need to determine how much energy can be extracted from a butane bottle. To do this, we must first calculate the amount of substance n of butane in the bottle. We can find this using the ideal gas law

$$pV = nRT \quad\Rightarrow\quad n = \frac{pV}{RT}\,,$$

where p is the pressure in the bottle, V is the volume of the bottle, T is the temperature, and R is the molar gas constant found in the List of Constants. From this, we can easily calculate the mass m using the molar mass M as

$$m = Mn\,.$$

Now, we calculate the energy E_{bottle} which we can obtain as

$$E_{\text{bottle}} = mH = HM\frac{pV}{RT} \doteq 6.6\,\mathrm{MJ}\,,$$

where H is the specific burning energy of butane.

The heat required to de-ice the pavement is given by the calorimetric equation

$$Q(S) = m_{\text{ice}}(S)\,c\Delta t + m_{\text{ice}}(S)\,l_{\text{t}}\,,$$

where c is the specific heat capacity of the ice, Δt is the temperature gain required to de-ice the ice, and l_t is the latent heat of fusion. The mass of the ice area m_{ice} (as a function of area) is then

$$m_{\mathrm{ice}}(S) = \rho_{\mathrm{ice}} d S \,,$$

where ρ_{ice} is the density of ice, d is the thickness of the ice, and S is its area. We now set up an equation between the energy in the butane bottle and the energy required to de-ice the pavement. We get

$$E_{\mathrm{bottle}} = Q(S)$$

$$HM\frac{pV}{RT} = \rho_{\mathrm{ice}} d S \left(c\Delta t + l_{\mathrm{t}}\right) \,.$$

From this, we can easily express the area of S as

$$S = \frac{HM}{\rho_{\mathrm{ice}} d}\frac{pV}{RT}\frac{1}{c\Delta t + l_{\mathrm{t}}} \doteq 2.1\,\mathrm{m}^2 \,.$$

Thus, we can see that this method of de-icing is highly inefficient, and other methods are more commonly used in practice.

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Problem CE ... wheelchair user

A wheelchair user can apply a force F to each wheel of the wheelchair. The mass of each wheel is m, their radius is R, and the mass of the wheelchair user and the rest of the wheelchair is M. What is the largest angle of an inclined plane the person can overcome using only their own power? Assume that the wheels do not slip on the ground and that the wheelchair cannot tip over. *Jarda pushed his grandmother uphill.*

When a wheelchair user stands on an inclined plane with an angle of inclination α, his center of mass is (in general) not located directly above the point of contact between the wheels and the plane. Therefore, since the wheels are connected only by a freely rotating axle, their weight force Mg acts vertically on this axle. The same applies to the weight of the rotating part of the wheelchair (the entire wheelchair except for the wheels). On the other hand, the wheels have their center of mass at their center, so their weight force is distributed also on their axle. In total, a force of $(M + 2m)g$ acts on the wheel axle, generating a torque of $(M + 2m)gR\sin\alpha$ with respect to the point of contact with the ground.

When the wheelchair user begins to apply a force F to the outer part of his wheel, he thereby develops a torque of magnitude FR with respect to the wheel axle. However, since the wheels do not skid and the inclined plane is stationary, this produces a total reaction torque of magnitude $2FR$ with respect to the point of contact, where the coefficient 2 represents the contribution from the two wheels. Thus, for the wheelchair to move upwards, it needs to satisfy the condition $2FR \geq (M + 2m)\, gR\sin\alpha$, which implies a boundary angle

$$\alpha = \arcsin\left(\frac{2F}{(M + 2m)g}\right) \,.$$

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Problem CF ... fast train in a turn

A high speed train travels on a track with a constant speed $v = 100\,\mathrm{km{\cdot}h^{-1}}$. It enters a curve with a radius $r = 1\,000\,\mathrm{m}$. By how many millimeters must the outer rail strip be higher than the inner rail strip, in order for the resultant force to act perpendicular to the plane of the tracks? The tracks have a gauge $d = 1\,435\,\mathrm{mm}$ and their angle of inclination with respect to the horizontal plane is small. *Prokop wanted to go home before Christmas...*

The turning train with a mass m is subject to a centrifugal force of magnitude

$$F_\mathrm{o} = \frac{mv^2}{r}\,,$$

which is perpendicular to the gravitational force $F_\mathrm{G} = mg$. The resulting force makes an angle of α with the force of gravity

$$\tan\alpha = \frac{F_\mathrm{o}}{F_\mathrm{G}}\,.$$

For the resultant force to act perpendicular to the plane of the track, the track must be tilted by an angle α to the horizontal plane

$$\sin\alpha = \frac{h}{d}\,,$$

where h is the desired track elevation and d is the track gauge. For small angles, we can write $\tan\alpha \approx \alpha \approx \sin\alpha$ and then

$$\frac{F_\mathrm{o}}{F_\mathrm{G}} = \frac{h}{d}\,,$$

from which we can express the height of the outer rail relative to the inner rail

$$h = \frac{F_\mathrm{o} d}{F_\mathrm{G}} = \frac{v^2 d}{rg} \doteq 113\,\mathrm{mm}\,.$$

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Problem CG ... ladybugs on a walk

Two ladybugs crawl up the arms of a right angle. On one of the arms, the first ladybug climbs toward the vertex from a distance of $l_0 = 30\,\mathrm{cm}$ at a speed of $u = 1.0\,\mathrm{cm{\cdot}s^{-1}}$. The second ladybug walks starting from the vertex along the other arm at a speed of $v = 1.5\,\mathrm{cm{\cdot}s^{-1}}$. How long after the start of the movement will the ladybugs be closest to each other?
Pepa shamelessly stole physics problems from freshman mechanics.

Let us denote the distance between the ladybugs as S and the distances of the ladybugs from the vertex as l and s. Since the ladybugs are moving along a right triangle, we can express S using the Pythagorean Theorem as

$$S = \sqrt{l^2 + s^2}\,.$$

The distances of the ladybugs from the vertex are here time-dependent, specifically as

$$l(t) = l_0 - ut\,,$$
$$s(t) = vt\,,$$

so the relation for S can be expanded as

$$S(t) = \sqrt{(l_0 - ut)^2 + v^2t^2}\,.$$

The minimum distance is found by finding the extremum of the function $S = S(t)$. By differentiating the function with respect to time, we get

$$\dot{S}(t) = \frac{v^2t + u(ut - l_0)}{\sqrt{(l_0 - ut)^2 + v^2t^2}}\,.$$

Now we set the derivative equal to zero

$$\dot{S}(t_0) = 0 \quad \Rightarrow \quad t_0 = \frac{ul_0}{u^2 + v^2}\,,$$

and by substituting the values from the assignment we obtain the solution [2]

$$t_0 \doteq 9.2\,\mathrm{s}\,.$$

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Problem CH ... flying octopus

Martin bought a plush octopus weighing $m = 90\,\mathrm{g}$ at a night market in the Philippines. However, he felt sorry for the octopus because it was so small that it could hardly see the wonders of the world. That's when he decided to expand its horizons. He took a broken shaft from a spring-loaded umbrella, which can be approximated as a spring with a stiffness constant $k = 25\,\mathrm{N{\cdot}m^{-1}}$. He put the octopus on top, compressed the shaft by length $d = 15\,\mathrm{cm}$ and released it so that the octopus flew straight up. By how many meters did the octopus's horizon expand at the highest point of its flight, compared to when it was just sitting on the uncompressed umbrella shaft? We are asking by how much the maximum distance, measured along the Earth's surface, increased, in one direction that the octopus could see. Assume the Earth to be a perfect sphere and that Martin held the umbrella shaft in such a way that when the shaft was compressed, the octopus was at a height $h_0 = 1.5\,\mathrm{m}$ above the Earth's surface.

Martin tried to entertain his octopus.

First, we determine the height the octopus will reach. Using the law of conservation of energy, where all the potential energy stored in the spring is converted into the gravitational potential energy of the octopus, we get

$$\frac{1}{2}kd^2 = E = mgh \quad \Rightarrow \quad h = \frac{kd^2}{2mg} \approx 31.9\,\mathrm{cm},.$$

The radial distance the octopus can see is given by $c = R_\mathrm{E}\varphi$, where R_E is the Earth's radius and φ is the angle between two nearly parallel lines passing through the Earth's center. One

[2] If we were to be totally correct, we should still need check that it is actually the minimum and not the maximum of the distance. We can verify this using for instance the second derivative of S. Alternatively, we can just look at the shape of S and notice that this function must necessarily decrease up to a certain t_0 and grow "to infinity" after this one is surpassed (but the relation holds only until the first ladybug reaches the vertex of the triangle). This implies that it must indeed be a minimum at t_0.

line corresponds to the octopus's trajectory, while the other extends to the farthest visible point on the Earth's surface. The angle φ can be determined using a right triangle as $\cos\varphi = \frac{R_\mathrm{E}}{H}$ with H representing the octopus's distance from the Earth's center at the peak of its flight.

As $H = R_\mathrm{E} + h + h_0$, we get

$$c = R_\mathrm{E} \arccos\left(\frac{R_\mathrm{E}}{H}\right) = R_\mathrm{E} \arccos\left(\frac{R_\mathrm{E}}{R_\mathrm{E} + h + h_0}\right) .$$

When the octopus is just sitting on the rod, the height is given by d, so

$$c_d = R_\mathrm{E} \arccos\left(\frac{R_\mathrm{E}}{R_\mathrm{E} + d + h_0}\right) .$$

Difference $\Delta c = c - c_d$ gives the distance the octopus can actually see

$$\Delta c = c - c_d = R_\mathrm{E}\left[\arccos\left(\frac{R_\mathrm{E}}{R_\mathrm{E} + h + h_0}\right) - \arccos\left(\frac{R_\mathrm{E}}{R_\mathrm{E} + d + h_0}\right)\right] \doteq 0.23\,\mathrm{km} .$$

The octopus's perception of the wonders will be expanded by about 230 meters.

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Problem DA ... pineapple on pizza reloaded

David threw two slices of pineapple with masses $m_1 = 0.30\,\mathrm{kg}$ *and* $m_2 = 0.60\,\mathrm{kg}$ *against each other over a pizza with radius* $16\,\mathrm{cm}$*. These two pineapples collided in a perfectly inelastic collision at a height of* $50\,\mathrm{cm}$ *above the center of the pizza, such that their vertical velocities were zero at the time of the collision. What is the highest velocity* v_2 *that the heavier slice can have before the collision if* $70\,\%$ *of energy was lost in the collision and both slices landed on the pizza after the collision? Furthermore, determine the speed* v_1 *that the lighter slice will have in this case. Consider that in a perfectly inelastic collision, the slices will stick together after colliding.* *David was promised canned pineapple for this problem.*

In order to find the highest velocity the heavier slice can move at for both slices to land on the pizza, we need to find the total velocity that the two combined pineapples can have. We find this by using the projectile motion equation for a height of $h = 0.5\,\mathrm{m}$ and the maximum distance they can to land at, $D = 0.16\,\mathrm{m}$

$$D = v \cdot \sqrt{\frac{2h}{g}} ,$$

from where we express v as

$$v = D \cdot \sqrt{\frac{g}{2h}} .$$

Since this is a perfectly inelastic collision (the slices stick together), the law of conservation of energy does not hold, but the conservation of momentum principle does. We therefore have the following equation for the speeds of the slices

$$v_1 \cdot m_1 - v_2 \cdot m_2 = -v \cdot (m_1 + m_2) .$$

In the equation we have introduced a sign convention where the direction of v_1 will be positive and the direction of v_2 will be negative. After the collision we expect the slices to move in the direction of v_2, therefore we will use the negative sign also on the right side of the equation. From this equation we can express v_1

$$v_1 = \frac{-\left(m_1 + m_2\right) \cdot v + m_2 \cdot v_2}{m_1} \,.$$

Let us express the loss of kinetic energy as

$$E_{\mathrm{k0}} - E_{\mathrm{k}} = 0.7 \cdot E_{\mathrm{k0}} \,,$$

where E_{k0} is the total kinetic energy of both slices before collision, i.e.,

$$E_{\mathrm{k0}} = \frac{1}{2} \cdot m_1 \cdot v_1^2 + \frac{1}{2} \cdot m_2 \cdot v_2^2 \,,$$

and E_{k} is the energy after the collision

$$E_{\mathrm{k}} = \frac{1}{2} \cdot \left(m_1 + m_2\right) \cdot v^2 \,.$$

After rearranging the equation, we get

$$0.3 \cdot E_{\mathrm{k0}} - E_{\mathrm{k}} = 0 \,.$$

By substituting in the speeds v and v_1, we obtain

$$\frac{3}{20} \cdot m_1 \cdot \left(\frac{-\left(m_1 + m_2\right) \cdot D\sqrt{\frac{g}{2h}} + m_2 \cdot v_2}{m_1}\right)^2 + \frac{3}{20} \cdot m_2 \cdot v_2^2 - \frac{1}{2} \cdot \left(m_1 + m_2\right) \cdot D^2 \cdot \frac{g}{2h} = 0 \,,$$

which after simplifying leads to a quadratic equation in its general form

$$v_2^2 \cdot \left(m_2 \cdot m_1 + m_2^2\right) - v_2 \cdot \left(2 \cdot m_2 \cdot D \cdot \sqrt{\frac{g}{2h}} \cdot \left(m_1 + m_2\right)\right) + \\ - \left(\frac{10}{3} \cdot \left(m_1 + m_2\right) \cdot D^2 \cdot \frac{g}{2h} \cdot m_1 - \left(m_1 + m_2\right)^2 \cdot D^2 \cdot \frac{g}{2h}\right) = 0 \,.$$

We consider the positive root to the equation, as negative speed (i.e. the magnitude of the velocity vector) is physically unrealistic. After substituting in the values, we see the maximum possible speed of the heavier slice is $v_2 \doteq 1.0\,\mathrm{m{\cdot}s^{-1}}$ and from the conservation of momentum principle we calculate the speed of the lighter slice to be $v_1 \doteq 0.58\,\mathrm{m{\cdot}s^{-1}}$.

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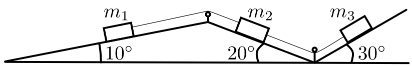
Problem DB ... hill climb racing reloaded



Let's have three inclined planes. The first one rises with slope $\alpha_1 = 10^\circ$, and after a sharp peak it flips into another one that falls with slope $\alpha_2 = 20^\circ$. After a sharp minimum, this flips into a plane rising with slope $\alpha_3 = 30^\circ$. On the respective planes, there are blocks with masses $m_1 = 20\,\mathrm{kg}, m_2 = 10\,\mathrm{kg}, m_3 = 15\,\mathrm{kg}$, which are connected by weightless ropes and pulleys (the ropes always find themselves parallel to the hill, see figure). With what magnitude of acceleration will a block of mass m_3 be moving? Do not consider friction. *A ridge trail was completed by Lego.*

The approach is completely analogous to the one in the problem *hill climb racing*. The only difference happens in the last step, where we plug the values into the same formulas we derived before. For the values used in this problem we get $a_3^{(\mathrm{tense})} \doteq 1.6\,\mathrm{m{\cdot}s^{-2}}$ and $a_3^{(\mathrm{loose})} \doteq 4.9\,\mathrm{m{\cdot}s^{-2}}$. The result is then determined from the condition for the tense string

$$a_3^{(\mathrm{tense})} < a_3^{(\mathrm{loose})} \quad \Rightarrow \quad a_3 = a_3^{(\mathrm{loose})} \doteq 4.9\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem DC ... an inclined cylinder on an inclined plane

A cylinder on an inclined plane is a typical textbook problem. But what if we incline the cylinder relative to the incline of the plane as well? We put a homogenous cylinder on an inclined plane facing the slope of the plane, but then we rotate it by 45° around the normal of the plane and let it go. What will be the magnitude of its velocity after time t? The plane is inclined by 30° relative to the horizontal direction. Assume that the cylinder is not slipping.
Things are going downhill for Jarda.

The cylinder will perform a rolling motion (rotating and also moving translationally), so its kinetic energy will consist of both a translational and a rotational component. The moment of inertia of a homogeneous cylinder of mass m and radius R is $I = mR^2/2$. If it rotates with angular velocity ω, its rotational kinetic energy is $I\omega^2/2 = mR^2\omega^2/4$. Since the cylinder does not slip, $v = R\omega$ holds for its velocity. Thus, if the cylinder rolls down the plane by some height Δh, it converts its potential energy to kinetic energy

$$mg\Delta h = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{3}{4}mv^2$$
$$\Longrightarrow \quad v = \sqrt{\frac{4}{3}g\Delta h}\,.$$

Since it does not slip on the plane, it will move along the line given by its initial orientation.

Let's denote the inclination of the plane as $\alpha = 30^\circ$ and the deflection of the cylinder on the inclined plane as $\beta = 45^\circ$. If we displace the cylinder in the vertical direction by δh, it will be $\delta h/\sin\alpha$ along the slope of the inclined plane. However, due to its trajectory being offset by an angle β, the total displacement will be $\Delta h(\sin\alpha\cos\beta)$. If we denote this path as l, the following equation holds for its velocity in this direction

$$v = \sqrt{\frac{4}{3}gl\sin\alpha\cos\beta} = \sqrt{\frac{\sqrt{2}}{3}gl}\,.$$

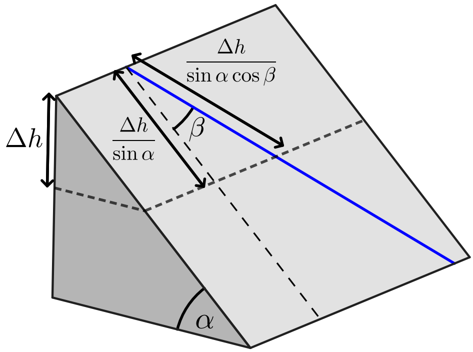

Figure 1: Sketch of the cylinder's trajectory.

We could solve the above differential equation for $l(t)$ to determine the velocity as a function of time. However, to simplify, we will use an analogy with the velocity relation for free fall. We only need to adjust the gravitational acceleration to $g' = (\sqrt{2}/6)g$. From this analogy, we get the dependence

$$v(t) = g't = \frac{2}{3}gt\sin\alpha\cos\beta = \frac{\sqrt{2}}{6}gt\,.$$

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Problem DD ... a well-fed spaceman

A spaceman lives on a base on the pole of planet Xeno. The planet's density is $\rho = 2\,707\,\mathrm{kg{\cdot}m^{-3}}$. Every morning, he weighs himself on his terrestrial scale and is very much concerned about the value shown. To make himself feel better, he moves to a base located on the equator of the planet, where he happily finds out he weighs 1.69 times less than on the pole. How long is a day on the planet? *Petr, on the other hand, would like to gain some weight.*

Let's denote m_p and m_r as the astronaut's weight at the pole and the equator of the planet. We know that

$$m_\mathrm{r} = \frac{m_\mathrm{p}}{1.69}\,,$$

so, the ratio between the weight at the pole and at the equator must be equal to

$$\frac{m_\mathrm{p}}{m_\mathrm{r}} = 1.69\,.$$

For the following calculations, it will be convenient to denote the inverse of this ratio as μ.

Since the weight of the spaceman is given as terrestrial, it is defined as the ratio of the gravitational force acting on him by the planet and the gravitational acceleration on Earth g.

$$m_\mathrm{p} = \frac{G\frac{mM}{R^2}}{g}\,,$$

where G is the gravitational constant, m is the astronaut's mass on Earth (inertial mass), and R is the radius of the planet Xeno.

The weight at the equator is calculated similarly, except that at the equator, the astronaut is also affected by a non-zero centrifugal force due to the planet's rotation. The magnitude of the centrifugal force is

$$F_\mathrm{o} = \omega^2 Rm\,,$$

and since the following relation

$$\omega = \frac{2\pi}{T}\,,$$

where T is the period of rotation (the length of one day) of the planet holds, we can write

$$F_\mathrm{o} = \frac{4\pi^2}{T^2} Rm\,.$$

The astronaut's weight at the equator is therefore

$$m_\mathrm{r} = \frac{G\frac{mM}{R^2} - \frac{4\pi^2}{T^2}Rm}{g}\,.$$

By dividing the equation by the first term on the right side, we get

$$\frac{m_\mathrm{r} g}{G\frac{mM}{R^2}} = 1 - \frac{4\pi^2 R^3}{T^2 GM}\,,$$

$$\mu = 1 - \frac{4\pi^2 R^3}{T^2 GM}\,.$$

The expression on the left side is the inverse of the ratio μ. If we solve for T^2, we obtain have

$$T^2 = \frac{1}{(1-\mu)}\frac{4\pi^2 R^3}{GM}\,.$$

However, we do not know the mass of the planet M or its radius R. Lucky for us, we can rewrite the equation as

$$T^2 = \frac{3\pi}{(1-\mu)G}\frac{\frac{4}{3}\pi R^3}{M}\,.$$

From this, it is easy to notice that we have the inverse of the density ρ on the right side. Therefore, we rewrite the equation as

$$T^2 = \frac{3\pi}{(1-\mu)G\rho}\,,$$

$$T = \sqrt{\frac{3\pi}{(1-\mu)G\rho}}\,,$$

and by substituting the values, we get the result

$$T \doteq 1.13 \cdot 10^4\,\mathrm{s} \doteq 3.14\,\mathrm{h}\,.$$

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Problem DE ... a wild carousel

Pepa placed a weight of mass m on a perfectly smooth horizontal plane and attached one end of a weightless ideal spring to it. He then took the other end in his hands and began to spin the weight in the horizontal plane at a constant angular velocity ω.

However, when the angular velocity exceeds a certain magnitude ω, it flies off to infinity. What is the maximum angular velocity with which the weight can rotate around the circle? The spring has stiffness k and rest length l_0. *Pepa was stealing from textbooks.*

If a body suspended on a spring moves in a plane along a circle with radius r, the elastic force of the spring F_p is equal (in size) to the centrifugal force F_o (from the point of view of the non-inertial system associated with the body)

$$F_\mathrm{p} = k(r - l_0) = m\omega^2 r = F_\mathrm{o}\,.$$

From this, we can easily express the radius of a circle as

$$r = \frac{l_0}{1 - \frac{m\omega^2}{k}}\,.$$

We can see that for a positive r, the expression in the denominator must also be positive, which implies a restriction on the ω in the form

$$1 - \frac{m\omega^2}{k} > 0 \qquad \Rightarrow \quad \omega < \sqrt{\frac{k}{m}} = \omega_\mathrm{max}\,.$$

For values of $\omega \to \omega_\mathrm{max}^+$ approaching the maximum from the right, we get $r \to +\infty$.

As a fun fact, we present a discussion of other cases. If the body is not rotating ($\omega = 0\,\mathrm{Hz}$), $r = l_0$ and r has the meaning of the length of the spring (not the radius of the circle, since the body does not rotate at all). We get the same result for an infinitely stiff spring $k \to +\infty$ as for any ω, we have $r = l_0$, so the spring behaves like a "rigid" rod.

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Problem DF ... P-roblematic oven

Anežka wanted to bake a cake for Jarda, so she needed to heat the oven to the temperature T_s. The thermometer in the oven compares the set temperature with the actual temperature inside T. A thermostat adjusts the heating power of the oven so that it is proportional to the difference of these two temperatures with the constant of proportionality K. If the temperature of both the room and the oven is T_a at the beginning of the heating process, at what temperature will the inside of the oven stabilize? The oven loses heat at a rate of $\kappa\,(T - T_\mathrm{a})$, and its heat capacity is C. *Jarda found out that the prices in Světozor confectionery have gone up!*

In many day-to-day systems, we need to set the value of some quantity. However, if the initial value of this quantity is different, we need to adjust it. These adjustments cannot be arbitrary, so a feedback loop is used to monitor the system's evolution and guide the process toward the target value. For instance, in temperature control, the goal is to regulate power to prevent overheating the oven unnecessarily or to minimize excessive fluctuations around the desired temperature. In our case, the power is proportional to the difference between the setpoint temperature T_s and the actual temperature T. This type of regulation is called *proportional* regulation. However, as we will see, this method has its limitations.

A more advanced approach is PID control, where the individual letters represent the proportional, integral, and derivative components. In this method, the control mechanism considers the current value of the quantity as well as its derivative and integral. By incorporating these factors, PID control allows for precise and smooth adjustment, enabling the system to reach the desired value without oscillations.

Let's return to solving our task. The amount of heat in the oven denoted as Q changes depending on how much heat is added by the external input P_in and how much heat is lost through thermal losses to the surroundings P_out. According to the problem, both of these quantities can be expressed as

$$P_\mathrm{in} = K\left(T_\mathrm{s} - T\right) , \quad P_\mathrm{out} = \kappa\left(T - T_\mathrm{a}\right) ,$$

and the heat of the oven is $Q = CT$. Therefore, we can write the heat balance in time t as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}T}{\mathrm{d}t} = P_\mathrm{in} - P_\mathrm{out} = K\left(T_\mathrm{s} - T\right) - \kappa\left(T - T_\mathrm{a}\right) = KT_\mathrm{s} + \kappa T_\mathrm{a} - \left(K + \kappa\right)T .$$

The task asks for the steady-state temperature. In this state, the temperature change with time is zero, which corresponds to a zero left-hand side of the equation. We can rearrange the right-hand side into the form

$$T_\mathrm{f} = \frac{KT_\mathrm{s} + \kappa T_\mathrm{a}}{K + \kappa} = T_\mathrm{s}\frac{1 + k\frac{T_\mathrm{a}}{T_\mathrm{s}}}{1 + k} ,$$

where T_f is the sought temperature and $k = \kappa/K$. If all the numbers are positive, we find that $T_\mathrm{f} < T_\mathrm{s}$, so the oven will never reach the desired temperature, which is a problem, to say the least. The only solution is either to minimize k or to implement a more sophisticated controller.

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Problem DG ... compressing air-filled syringe

Lego was playing with a syringe. He filled it to a volume of $V = 20\,\mathrm{ml}$, then he blocked the opening and began compressing it very slowly, so that the air inside had time to exchange heat with the surroundings through the walls. How much work did Lego perform when he compressed the syringe to a volume of $V/2$? The surroundings were under normal conditions the entire time. *Lego thought, that many people forget about surrounding air.*

The formula for the work of an isothermal process is

$$W = nRT\ln\frac{V_2}{V_1} ,$$

where V_2 is the final volume and V_1 is the initial volume. However, we have to remember that W is the work done by the gas. We are interested in the work done by us on the gas, which has the opposite sign, thus inverting the fraction in the logarithm.

The gas in the syringe had a volume V and a pressure p_a at the beginning, so we can express from the equation of state

$$p_\mathrm{a}V = nRT\,,$$

which, when substituted into the equation for work, yields

$$W = p_\mathrm{a}V\ln 2 \doteq 1.4\,\mathrm{J}\,.$$

It is intuitive that if I squeeze the syringe to half its original volume, I am realistically doing less work than if I, for example, lift a 1.5 L bottle (about 15 N) 10 centimeters up (which is only a little more than 1.4 J). So where did we make an “error”? We forgot to account for the action of the air we have around us. It helps us since it compresses the syringe together with us. We therefore don’t have to overcome all the pressure in the syringe, just the difference between it and the atmospheric pressure. Since the atmospheric pressure remains constant while the syringe is being squeezed, we can easily calculate how much work has been done by the surrounding atmosphere to squeeze the syringe

$$W_\mathrm{A} = p_\mathrm{a}\Delta V = p_\mathrm{a}\frac{V}{2} \doteq 1.0\,\mathrm{J}\,.$$

We can calculate the work to be done by Lego as the difference between these two works

$$W = W_\mathrm{A} + W_\mathrm{L} \quad\Rightarrow\quad W_\mathrm{L} = W - W_\mathrm{A} = p_\mathrm{a}V\left(\ln 2 - \frac{1}{2}\right) \doteq 0.39\,\mathrm{J}\,.$$

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Problem DH ... a stretching suspense

Monča placed an apple on the edge of a spinning carousel and then stepped back. Standing at the same height as the carousel, she stretched a rubber band weighing 5.0 g and shot it towards the center of the carousel at a 45° angle to the horizontal, successfully knocking the apple down. Determine the lowest possible circumferential velocity of the carousel if you know that the rubber band has a stiffness of $10\,\mathrm{N\cdot m^{-1}}$ and Monča would have succeeded in knocking the apple if she stretched it to either 6.0 cm or 5.0 cm. *Monča played on the playground.*

We will solve the problem using the law of conservation of energy. First, we express the energy of a stretched rubber band just before it flies off to the carousel. We approximate the rubber band as a perfect spring, so its energy is given by

$$E = \frac{1}{2}ky^2\,,$$

where k is the stiffness of the rubber band, and y is the distance from the equilibrium position to which it was stretched. After releasing the rubber band, all the energy is converted into kinetic energy, which gives us

$$E = E_\mathrm{k} = \frac{1}{2}ky^2 = \frac{1}{2}mv^2\,.$$

From this, we can express the velocity of the rubber band as

$$v = \sqrt{\frac{ky^2}{m}}\,.$$

To calculate the tangential velocity, we need the difference in distances between the two shots and the difference in their flight times. The time difference gives us half of the carousel's period since one shot occurs when the apple is closer to Monča and the other when it is on the opposite side – meaning the rubber band must travel across the entire carousel. We derive

$$v_{\mathrm{kol}} = \frac{2\pi r}{T} = \frac{2\pi \frac{d}{2}}{2\frac{T}{2}} = \frac{\pi d}{2\Delta t} = \frac{\pi(x_1 - x_2)}{2(t_1 - t_2)}\,,$$

where r is the radius of the carousel, d is its diameter, x_1, x_2 are the distances the rubber band flies to, and t_1, t_2 are the flight times of the rubber band.

Let us analyze the projectile motion itself. As the band flies off at 45°, the horizontal and vertical components of the velocity are the same. From the right triangle, we know that

$$v = \sqrt{v_x^2 + v_y^2}\,,$$

and we can also express the velocity as

$$v = \sqrt{2v_x^2} = \sqrt{2}v_x\,.$$

From the equations for projectile motion, we get two equations. One for the horizontal distance and the other for the height

$$x = v_x t\,,$$
$$0 = v_x t - \frac{1}{2}gt^2\,.$$

From the first equation, we express the time and substitute it into the second equation, which gives us

$$x = \frac{2v_x^2}{g}\,.$$

To express the time, we will use the first of the two equations to obtain

$$t = \frac{x}{v_x} = \frac{2v_x}{g}\,.$$

Now, we need to substitute the velocity from the law of conservation of energy

$$v_x = \frac{v}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{ky^2}{m}}\,.$$

Finally, we perform all the necessary substitutions and obtain the difference in distances

$$(x_1 - x_2) = \frac{2v_{x1}^2}{g} - \frac{2v_{x2}^2}{g} = \frac{ky_1^2}{mg} - \frac{ky_2^2}{mg} = \frac{k}{mg}\left(y_1^2 - y_2^2\right)\,,$$

and the difference in times

$$(t_1 - t_2) = \frac{2v_{x1}}{g} - \frac{2v_{x2}}{g} = \sqrt{\frac{2k}{mg^2}}y_1 - \sqrt{\frac{2k}{mg^2}}y_2 = \sqrt{\frac{2k}{mg^2}}\left(y_1 - y_2\right)\,.$$

Now we substitute into the equation for velocity

$$v_{\text{kol}} = \frac{\pi(x_1 - x_2)}{2(t_1 - t_2)} = \frac{\pi \frac{k}{mg}\left(y_1^2 - y_2^2\right)}{2\sqrt{\frac{2k}{mg^2}}\,(y_1 - y_2)} = \pi\sqrt{\frac{k}{8m}}\frac{(y_1 + y_2)(y_1 - y_2)}{(y_1 - y_2)} = \pi\sqrt{\frac{k}{8m}}\,(y_1 + y_2)\ ,$$

and after substituting the values from the task, we get

$$v_{\text{kol}} = \pi\sqrt{\frac{k}{8m}}\,(y_1 + y_2) = \pi\sqrt{\frac{10}{8 \cdot 0.005}}\,(0.06 + 0.05) \doteq 5.5\,\mathrm{m\cdot s^{-1}}\,.$$

The tangential velocity of the carousel is $5.5\,\mathrm{m\cdot s^{-1}}$.

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Problem EA ... Fingal's Fingers

The strongman is lifting a huge and certainly very heavy rod. This homogeneous rod has a mass $m = 250\,\mathrm{kg}$ and a length $l = 4.5\,\mathrm{m}$. One end of the rod is fixed to the ground, and it can rotate freely around this point. The strongman lifts it by starting at the opposite end, always holding the rod at a height $h = 180\,\mathrm{cm}$, and moving closer to the fixed point as he goes. What is the minimum force he needs to exert on the rod when his distance from the attachment point is $x = 1.5\,\mathrm{m}$? Assume the strongman is standing upright. *Lego was doomscrolling...*

We can calculate the minimum required force by analyzing the torque equilibrium. We will compute these torques relative to the point where the rod is attached, since it would be difficult to calculate the force acting inside the rod itself.

The weight acting on the rod has a magnitude $F_\mathrm{g} = mg$ and acts at its center of mass, which is located at the midpoint of the rod. Since this force acts vertically downward, we can obtain its torque by multiplying it by the horizontal distance between the attachment point and the center of the bar. The angle between the rod and the ground can be found using the strongman's height and distance with the formula:

$$\varphi = \arctan\frac{h}{x}\,.$$

The horizontal distance from the center of the rod to the axis of rotation will be $x_{l/2} = (l/2)\cos\varphi$, so the resulting torque due to grafity is

$$M_\mathrm{g} = F_\mathrm{g} x_{l/2} = mg\frac{l}{2}\cos\left(\arctan\frac{h}{x}\right)\,.$$

Now, let's say the strongman pushes with a force F (the quantity we are solving for). What is the length of the lever arm? Since the strongman is not limited to pushing only straight up, and we are interested in the minimum force require, we are actually looking at the situation where he pushes perpendicular to the lever arm. The length of this arm can be easily calculated using the Pythagorean theorem: $r = \sqrt{x^2 + h^2}$. The torque will therefore be equal to

$$M_\mathrm{s} = F\sqrt{x^2 + h^2}\,.$$

We still need to get rid of the term $\cos\left(\arctan(h/x)\right)$ in M_g. We can either use some trigonometric identities, or notice that the triangle with hypotenuse $l/2$, which we wanted to calculate the adjacent side for, is similar to the triangle we've already used (with a hypotenuse of $\sqrt{x^2+h^2}$ and adjacent side x). So instead of using trigonometry, we can directly write

$$x_{l/2} = \frac{l}{2}\frac{x}{\sqrt{x^2+h^2}}\,.$$

Now we equate the torques

$$M_\mathrm{g} = M_\mathrm{s}\,,$$

$$mg\frac{l}{2}\frac{x}{\sqrt{x^2+h^2}} = F\sqrt{x^2+h^2}\,,$$

$$F = \frac{1}{2}mg\frac{lx}{x^2+h^2} \doteq 1.5\,\mathrm{kN}\,.$$

From this equation, we can see that when the strongman is near the end of the rod, he will need to apply a very small amount of force. However, when $x \gg h$, the required force will be proportional to $1/x$.

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Problem EB ... Lego borrowed Verča's coin

Lego borrowed Verča's coin from last year and decided to explore its properties. He turned on a moving conveyor belt, which reached a velocity v. Then, he took the coin and placed it on the belt in such a way that it could roll in the direction of the belt's motion and that at the moment of release, it was neither moving nor rotating. The coefficient of kinetic friction between the coin and the belt is f. How long will it take for the coin to stop slipping? The coin is a homogeneous disk with a radius r and mass m.

... and had Lego done it earlier, he could have advised Vojta with pétanque...

What happens when we place a coin on a moving belt? It will slip, and thus, the frictional force will cause it to rotate and accelerate. This slipping stops when the velocity of the point at the bottom of the coin reaches the velocity of the moving belt v. The velocity of the bottom point is equal to the sum of the velocity of motion of the center of the coin and the circumferential velocity of rotation of the coin. Let's discuss the time evolution of both of these velocities.

We start with the acceleration of the center of the coin. This part is straightforward – the force acting on the coin is simply the frictional force. Since friction is the product of the coefficient of friction and the normal force, and in this case, the normal force equals the gravitational force, we get

$$F_\mathrm{t} = fmg.$$

We can obtain the acceleration by dividing this equation by the mass of the coin, so $a = F_\mathrm{t}/m = fg$. Furthermore, the translational velocity of the coin is $v_\mathrm{t}(t) = fgt$.

To find the rotational velocity, we first need to find the angular acceleration, which we get as the ratio of the torque and the moment of inertia. We already know that the frictional force acting on the coin is $F_\mathrm{t} = fmg$. Since the coin has radius r, the torque with respect to the center of the coin will be $M_\mathrm{t} = F_\mathrm{t} r = fmgr$. The moment of inertia of a homogeneous disk of mass m and radius r is $I = mr^2/2$. Thus, the angular acceleration of the coin is

$$\varepsilon = \frac{M_\mathrm{t}}{I} = \frac{fmgr}{\frac{1}{2}mr^2} = \frac{2fg}{r}\,.$$

The angular velocity of the coin's rotation will vary in time as $\omega(t) = \varepsilon t = 2fgt/r$, and the circumferential velocity will therefore be $v_\mathrm{r}(t) = \omega(t)r = 2fgt$.

It remains to think about the direction in which the moving belt will make the coin move and rotate. At the bottom of the coin, the translational and rotational components of the velocity have the same direction. Therefore, the resulting speed of the point at the bottom of the coin will be the sum of the magnitudes of these two velocities $v_\mathrm{v}(t) = v_\mathrm{t}(t) + v_\mathrm{r}(t) = 3fgt$.

We need to determine when the slipping stops. As mentioned earlier, this occurs when the velocity v_v of the coin's bottom point matches the belt's velocity v

$$3fgt = v \quad \Rightarrow \quad t = \frac{v}{3fg}\,.$$

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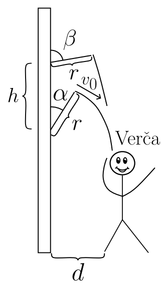
Problem EC ... showered

Verča is showering under a stream of water coming out of a shower head mounted on a stand. The shower head has a length $r = 25\,\mathrm{cm}$ and is inclined at an angle $\alpha = 45^\circ$ to the vertical axis of the stand. Water is splashing out perpendicularly at a velocity of $v_0 = 5.5\,\mathrm{m\cdot s^{-1}}$. The water jet hits Verča at a horizontal distance $d = 50\,\mathrm{cm}$ from the stand. However, she prefers the water to hit her from a higher height, so she moves the shower head up by h. She then adjusts the angle between the head and the stand to $\beta = 60^\circ$ so that the water will again fall on her at a distance of d. By how much did Verča move the head up on the stand? Ignore the thickness of the water stream and consider that the head, the stand, and the point where the water hits Verča are in the same plane.



Ideas often come in a shower.

The water starts spraying from a distance $d_0 = r\sin\alpha$ from the stand axis. The projection of its velocity into the horizontal direction will be $v_{0x} = v_0\cos\alpha$. From this, we can calculate how much time the water spends traveling from the showerhead to Verča

$$t_1 = \frac{d - d_0}{v_{0x}} = \frac{d - r\sin\alpha}{v_0\cos\alpha}\,.$$

From this, we can calculate the height difference h_1 between the point where water exits the head and where it falls on Verča

$$h_1 = v_{0y}t_1 + \frac{1}{2}gt_1^2 = v_0\sin\alpha\frac{d - r\sin\alpha}{v_0\cos\alpha} + \frac{1}{2}g\left(\frac{d - r\sin\alpha}{v_0\cos\alpha}\right)^2 \doteq 36\,\mathrm{cm}\,,$$

where g is the gravitational acceleration.

Afterward, we can repeat this with the new angle β, from which we get the height difference h_2 for this case. Let's not forget that the height difference is caused by two factors:

Verča's adjustment of the showerhead's height and the change in the head's angle. To calculate the head's adjustment in height, we must subtract the change caused by the adjustment of the head's angle. The difference in height between the mount point and head is $r\cos\alpha$ (the mount point is lower by this amount), so $h_1 - r\cos\alpha$ is by h lower than $h_2 - r\cos\beta$. Therefore, we get this equation

$$\sin\alpha\frac{d - r\sin\alpha}{\cos\alpha} + \frac{1}{2}g\left(\frac{d - r\sin\alpha}{v_0\cos\alpha}\right)^2 - r\cos\alpha + h =$$
$$= \sin\beta\frac{d - r\sin\beta}{\cos\beta} + \frac{1}{2}g\left(\frac{d - r\sin\beta}{v_0\cos\beta}\right)^2 - r\cos\beta$$

$$h = \frac{1}{2}g\left(\left(\frac{d - r\sin\beta}{v_0\cos\beta}\right)^2 - \left(\frac{d - r\sin\alpha}{v_0\cos\alpha}\right)^2\right) +$$
$$+ \sin\beta\frac{d - r\sin\beta}{\cos\beta} - \sin\alpha\frac{d - r\sin\alpha}{\cos\alpha} - r(\cos\beta - \cos\alpha) \doteq 24\,\mathrm{cm}\,.$$

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Problem ED ... journey to the off-center of the Earth

What is the angle between the line connecting the center of the Earth to a point on the $\varphi = 50.1^\circ$ latitude on the Earth's surface and the direction of the gravity of Earth at that point?

Lego really thinks about questions like this one.

The weight acting on a mass point on the surface of the Earth is the sum of the gravitational force, which pulls the mass point toward the Earth's center, and the centrifugal force, which (seemingly) acts on it due to its rotation with the Earth's surface. This force acts in a direction perpendicular to the Earth's axis and lies in the plane defined by that axis and the given mass point. The "weight" acceleration is then the sum of the gravitational acceleration and the centrifugal acceleration. We can obtain both by dividing the force acting on the mass point by its mass m. Let us first calculate the magnitudes of these accelerations.

The Earth has a mass M and a radius R, so the gravitational acceleration will have a magnitude of

$$a_G = G\frac{M}{R^2} \doteq 9.80\,\mathrm{m{\cdot}s^{-2}}\,,$$

where G is the universal gravitational constant.

For an angle φ from the equator, the radius of rotation is $r = R\cos\varphi$. Furthermore, we know that the Earth completes one rotation every $T = 24\,\mathrm{h}$ hours, so its angular frequency of rotation is $\omega = 2\pi f = 2\pi/T$. The magnitude of the centrifugal acceleration is then

$$a_{\mathrm{od}} = \omega^2 r \doteq 0.0216\,\mathrm{m{\cdot}s^{-2}}\,.$$

Now, we move on to the trigonometry. Both accelerations lie in a plane defined by the Earth's axis and the point where they act. The centrifugal acceleration is perpendicular to the Earth's axis and points outward, away from the axis. The gravitational acceleration has two components: one perpendicular to the axis, with magnitude $a_G\cos\varphi$, directed inward toward the axis, and one parallel to the axis, with magnitude $a_G \sin\varphi$, directed toward the center of the Earth. The weight acceleration (the one that comes from the sum of the previous two) will have a component parallel to the Earth's axis equal to the gravitational acceleration, and a component perpendicular to the Earth's axis with magnitude $a_G \cos\varphi - a_{\text{od}}$. The angle that the gravitational acceleration makes with the perpendicular to the Earth's axis will then be

$$\alpha = \arctan \frac{a_G \sin\varphi}{a_G \cos\varphi - a_{\text{od}}} = 50.2^\circ\,.$$

Latitudes are most commonly measured from the perpendicular to the Earth's axis (the equator is at 0° and the poles at 90°). So, to determine the angle between the line connecting the point and the center of the Earth and the gravitational acceleration, we must take the difference between them. In our case, $|\alpha - \varphi| \approx 0.1^\circ$.

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Problem EE ... conductive air

A parallel plate capacitor with an area A and spacing d is charged to voltage U. Determine how long it takes for this voltage to drop to one third if there is air of conductivity σ between the plates. *Jarda's batteries are dead.*

The capacitance of a capacitor is typically determined as

$$C = \frac{\varepsilon A}{d}\,,$$

where ε represents the permittivity, and the meaning of A and d is given in the problem. The voltage U and the charge on the plates Q are related by the equation $Q = CU$.

Let us assume that the air between the plates forms a resistor with a large resistance, specifically

$$R = \frac{1}{\sigma}\frac{d}{A}\,.$$

We can determine the displacement current between the plates as

$$I = \frac{U}{R}\,,$$

and the charge on the plates decreases as

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = -C\frac{\mathrm{d}U}{\mathrm{d}t} = I = \frac{U}{R}\,.$$

We can solve this differential equation by a method called separation of variables

$$U = U_0 \exp\left(-\frac{t}{CR}\right)\,.$$

Now, we need to substitute the previously found values of resistance and capacitance

$$CR = \frac{\varepsilon A}{d}\frac{1}{\sigma}\frac{d}{A} = \frac{\varepsilon}{\sigma}\,.$$

The time evolution of the voltage across the capacitor is

$$U = U_0 \exp\left(-\frac{\sigma t}{\varepsilon}\right) .$$

Therefore, the final solution is

$$T = \frac{\varepsilon}{\sigma} \ln(3) .$$

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Problem EF ... my brother Coriolis

Pepa wanted a practical reminder of gravity, so he came up with a certain (thought) experiment: by what distance from the base of a perpendicular does a textbook deviate during a free fall from the Institute of Theoretical Physics at MFF UK due to the Coriolis force? Consider that the Institute is at height $h = 60\,\mathrm{m}$, the air resistance is negligible and the latitude of Prague is $\varphi = 50^\circ$ N. The relation for Coriolis acceleration is $\vec{a}_\mathrm{C} = -2\,\vec{\omega} \times \vec{v}$.

Pepa had an exam on general relativity.

We will neglect higher-order effects in our solution and assume simple free fall (uniformly accelerated motion), with the addition of the Coriolis force acting on the body. Since this force is always perpendicular to the velocity $\vec{v}$ of the falling object, as given in the problem statement, and the velocity remains nearly vertical, we will only consider its effect in the horizontal direction.

In our case, the Coriolis force will be due to the Earth's rotation about its axis, so our angular velocity magnitude will be $\omega = 1\,\mathrm{rot{\cdot}day^{-1}} \doteq 7.27 \cdot 10^{-5}\,\mathrm{rad{\cdot}s^{-1}}$ and its direction will be identical to the direction of the Earth's axis of rotation. Hence, the angle between the vectors $\vec{\omega}$ and $-\vec{v}$ will be $90^\circ - \varphi$, so

$$a_\mathrm{C} = -2\,|\vec{\omega} \times \vec{v}| = 2\,|\vec{\omega}|\,|\vec{v}| \sin(90^\circ - \varphi) = 2\omega v \cos(\varphi) .$$

By integrating this acceleration over time, we obtain the value of the horizontal component of the velocity $v_\mathrm{v}(t)$ of the body at time t. We use the approximation that the horizontal component remains significantly smaller than the vertical component of velocity v_z throughout the fall, so $|\vec{v}(t)| = \sqrt{v_\mathrm{z}^2(t) + v_\mathrm{v}^2(t)} \approx v_\mathrm{z}(t) = gt$. We then have

$$v_\mathrm{v}(t) = \int_0^t a_\mathrm{C}(t)\;\mathrm{d}t = \int_0^t 2\omega g t \cos(\varphi)\;\mathrm{d}t = \omega g t^2 \cos(\varphi) ,$$

where we set the initial value of the horizontal velocity $v_\mathrm{v}(0)$ to zero (as well as $v_\mathrm{z}(0)$). Further integration over time gives the trajectory that the body travels in the horizontal direction, i.e., exactly the deviation from the straight vertical trajectory we are looking for

$$s_\mathrm{v}(t) = \int_0^t v_\mathrm{v}(t)\;\mathrm{d}t = \int_0^t \omega g t^2 \cos(\varphi)\;\mathrm{d}t = \omega g \frac{t^3}{3} \cos(\varphi) .$$

Now, we need to substitute the time it takes the body to fall freely (in the vertical direction only) from the height h, so

$$h = g\frac{t_1^2}{2} \implies t_0 = \sqrt{\frac{2h}{g}} .$$

Thus, the deviation we are looking for is

$$s_\mathrm{v}(t_0) = \frac{2}{3}\sqrt{\frac{2h^3}{g}}\omega\cos(\varphi) \doteq 6.5\,\mathrm{mm}\,.$$

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Problem EG ... interstellar chocolate

An unnamed chocolate company decided to distribute chocolate to distant planets. Calculate the longest distance they can send a shipment of ten 100-gram bars that expire in exactly $t_\mathrm{cok} = 1.0\,\mathrm{y}$ *and about which the company can tell the aliens that their* $m_\mathrm{dod} = 5.0\,\mathrm{kg}$ *of chocolate is on its way. Consider that a speed approaching the speed of light has no effect on the properties of the chocolate (taste, smell, expiration date etc...), and that the time it takes for the chocolate to accelerate is negligible relative to the time of the entire flight.*

...weren't the chocolates bigger?

To calculate the farthest possible distance, we need to calculate the speed the delivery must travel at and the time the chocolate can fly before it expires.

First, we calculate the speed at which the condition for the weight of the chocolate delivery is met. This speed is simply expressed from the formula for the relativistic mass as

$$m_\mathrm{dod} = \frac{m_\mathrm{cok}}{\sqrt{1-\frac{v^2}{c^2}}}\,.$$

So the speed will be

$$v = c\sqrt{1-\left(\frac{m_\mathrm{cok}}{m_\mathrm{dod}}\right)^2}\,.$$

Now let's calculate the time. Consider that while the chocolate is moving almost at the speed of light, this will have no effect on its properties, i.e. its expiration time will remain the same. However, we also know that at this speed time passes at different rate for the chocolate, so we must recalculate the maximum amount of time that chocolate can fly for time in its system to be exactly one year. We use the formula for the time dilation

$$t_\mathrm{cok} = t\sqrt{1-\frac{v^2}{c^2}}\,,$$

$$t = \frac{t_\mathrm{cok}}{\sqrt{1-\frac{v^2}{c^2}}}\,.$$

Now we will substitute the calculated speed into the expressed time to get

$$t = \frac{t_\mathrm{cok}}{\sqrt{1-\frac{c^2\left(1-\left(\frac{m_\mathrm{cok}}{m_\mathrm{dod}}\right)^2\right)}{c^2}}} = \frac{t_\mathrm{cok}m_\mathrm{dod}}{m_\mathrm{cok}}\,.$$

All we have to do now is substitute the two calculated maxima into the formula for the distance

$$x = tv = \frac{t_{\mathrm{cok}} m_{\mathrm{dod}}}{m_{\mathrm{cok}}} c \sqrt{1 - \left(\frac{m_{\mathrm{cok}}}{m_{\mathrm{dod}}}\right)^2} = t_{\mathrm{cok}} c \sqrt{\left(\frac{m_{\mathrm{dod}}}{m_{\mathrm{cok}}}\right)^2 - 1}\,.$$

Finally, we just substitute the numerical values from the assignment. We can notice that the product $t_{\mathrm{cok}} \cdot c$ gives us the light year, so we just need to evaluate the expression in the square root

$$x = \left(1 \sqrt{\left(\frac{5.0\,\mathrm{kg}}{1.0\,\mathrm{kg}}\right)^2 - 1}\right) \mathrm{ly} = \left(1 \cdot \sqrt{24}\right) \mathrm{ly} \doteq 4.9\,\mathrm{ly}\,.$$

The farthest we can send this shipment of chocolate is 4.9 light years.

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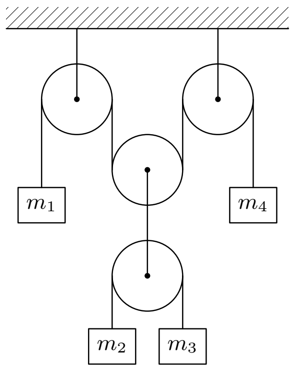
Problem EH ... 4 pulleys and 4 weights



Let us consider pulleys and weights arranged as shown in the figure. The weights have masses of $m_1 = 1.1\,\mathrm{kg}$, $m_2 = 2.2\,\mathrm{kg}$, $m_3 = 3.3\,\mathrm{kg}$ and $m_4 = 4.4\,\mathrm{kg}$. All ropes and pulleys are massless and perfectly inextensible. What is the acceleration of the weight with mass m_1? Use a positive sign if it accelerates downward and a negative sign if it accelerates upward.

Lego wanted a fairly typical pulley problem.

The fact that the ropes and pulleys are massless means that the resulting forces and torques acting on them must be zero (because $F = ma = 0a = 0$). In the following paragraph, we will repeatedly use this fact.

For both ropes on which the weights hang, the tension is the same throughout the length of each rope. Let us denote the tensions in the ropes as T_1 and T_2. The upper free pulley is pulled upward by a force of $2T_1$, so the rope hanging from it must pull it downward with the same force. This rope is also massless and is thus pulling the lower free pulley upward with a force of $2T_1$. This pulley is pulled downward by the second rope with a force of $2T_2$. Therefore, it must hold that $2T_1 = 2T_2$, from which it is clear that the tensions in both ropes are equal. Let us denote $T_1 = T_2 = T$.

Let us define the acceleration of each weight as positive if it points downward and negative if it points upward. We can then write the equations of motion (Newton's second law) for all the weights as follows

$$m_1 a_1 = m_1 g - T\,,$$
$$m_2 a_2 = m_2 g - T\,,$$
$$m_3 a_3 = m_3 g - T\,,$$
$$m_4 a_4 = m_4 g - T\,.$$

It is also necessary to use the fact that the ropes do not change their length (and the pulleys retain their shape). What would happen if we slightly moved the weights? The upper free pulley moves upward by the average displacement of weights m_1 and m_2 downward. The lower pulley must move upward by the same distance (the average displacement of weights m_1 and m_2), so the center of mass of weights m_3 and m_4 must also move upward by the same distance. That gives us the condition for the displacements of the weights. Let us denote the displacement of a weight as positive if it moves downward and negative if it moves upward. We can express this condition mathematically as

$$\frac{\Delta x_1 + \Delta x_2}{2} = -\frac{\Delta x_3 + \Delta x_4}{2}\,,$$
$$\Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4 = 0\,.$$

When we take the second derivative of this relation with respect to time, we get the condition for the accelerations

$$a_1 + a_2 + a_3 + a_4 = 0\,.$$

Thus, we have 5 equations (4 motion equations and 1 condition for accelerations) and 5 unknowns (4 accelerations and 1 tension in the rope). Let us solve them. From the motion equations, we substitute all the accelerations except for a_1 (since that is the one we are interested in) into the condition for accelerations

$$a_1 + 3g - T\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}\right) = 0\,.$$

Now, substitute for T from the first equation of motion

$$a_1 + 3g + m_1(a_1 - g)\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}\right) = 0\,.$$

We divide everything by m_1 and rearrange the equation, isolating the terms with a_1 on one side and the terms with g on the other. Finally, we divide by the factor in front of a_1 and get the result

$$a_1 = \frac{-\frac{3}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}}{\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}}\, g \doteq -9.0\,\mathrm{m\cdot s^{-2}}\,.$$

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Problem FA ... Three Gorges Dam

How did the length of the day change after the Three Gorges Dam in China was filled? The maximum volume of the dam is $39.3\,\mathrm{km^3}$*, its average altitude is* $160\,\mathrm{m}$ *above the sea level, and its coordinates are* 30° *N and* 111° *E. Assume that the water was originally distributed uniformly over the whole surface of Earth.* *And yet people can move the Earth!*

Consider a model where water from the ocean surface is transferred to a single point to fill the given volume of the dam. This changes the mass distribution over the entire planet, its moment of inertia, and, according to the law of conservation of angular momentum, its rotational velocity, and the length of the day. The altitude is negligible compared to the Earth's radius, so we consider the Three Gorges Dam to be at one point on the Earth's surface.

The initial moment of inertia is

$$J_i = \frac{2}{5} M_\oplus R_\oplus^2 \,.$$

The volume of water transferred is $V = 4\pi R_\oplus^2 d$, where $d \doteq 0.08\,\mathrm{mm}$ is the thickness of the water at the surface that is being removed to the Three Gorges Dam. The Earth's moment of inertia, ignoring this surface layer is

$$J_{-} = J_i - \frac{2}{3} V \rho R_\oplus^2 \,,$$

where ρ is the density of water. We can neglect the thickness of the water layer in the equation since $d \ll R_\oplus$. The water in the dam is now treated as being concentrated at a single point, located at a distance $R_\oplus \cos\varphi$ from Earth's axis of rotation, where $\varphi \doteq 30^\circ$ represents the dam's latitude. The new moment of inertia of the Earth is therefore

$$J_\mathrm{f} = J_\mathrm{i} - \frac{2}{3} V \rho R_\oplus^2 + V \rho R_\oplus^2 \cos^2\varphi \,.$$

Now, let us use the law of conservation of angular momentum

$$J_\mathrm{i}\omega_\mathrm{i} = J_\mathrm{f}\omega_\mathrm{f} \,,$$

where

$$\omega = \frac{2\pi}{T}$$

is the angular velocity of the Earth's rotation, and T is its period. We get

$$\frac{J_\mathrm{i}}{T_\mathrm{i}} = \frac{J_\mathrm{f}}{T_\mathrm{f}} \quad \Rightarrow \quad T_\mathrm{f} = T_\mathrm{i} \frac{J_\mathrm{i} - \frac{2}{3} V \rho R_\oplus^2 + V \rho R_\oplus^2 \cos^2\varphi}{J_\mathrm{i}} = T_\mathrm{i} \left(1 - \frac{5 V \rho}{2 M_\oplus} \left(\frac{2}{3} - \cos^2\varphi\right)\right) \,.$$

The change in the length of a day is therefore

$$\Delta T = T_\mathrm{f} - T_\mathrm{i} = T_\mathrm{i} \frac{5 V \rho}{2 M_\oplus} \left(\cos^2\varphi - \frac{2}{3}\right) \doteq 0.12\,\mathrm{\mu s} \,.$$

The day has lengthened by $0.12\,\mathrm{\mu s}$.

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Problem FB ... TEM

It is possible to observe the diffraction of electrons on the periodic lattice of atoms in crystals using the transmission electron microscope (TEM). Assume that electrons accelerated by the voltage of $90.0\,\mathrm{kV}$ *hit perpendicularly a square lattice of gold atoms with a lattice constant of* $407.856\,\mathrm{pm}$. *How many degrees are the electrons deflected at the first-order maximum?*

Jarda wanted to combine all areas of physics into one problem.

To solve this problem, we must use our knowledge of quantum physics, relativity, and wave optics. Electrons interacting with the atomic lattice exhibit wave properties determined by their wavelength, which in turn depends on their momentum. We must use special relativity to accurately determine this momentum, as classical physics would not provide the correct result.

Finally, using the knowledge of the wavelength of electrons, we calculate their deviation from the original direction.

Starting from the end – the equation for the diffraction of a wave on a grid is

$$d \sin\alpha = k\ \lambda\,,$$

where d is the distance of the atoms in the lattice, α is the angle from the normal to the surface, $k = 1$ is the order of the first maximum, and λ is the wavelength of the incident electrons. This relationship can be derived through simple reasoning. The strongest electron scattering occurs in directions where constructive interference takes place. For this to happen, the difference in the distances traveled by the electrons must be an integer multiple of the wavelength λ. That forms the right-hand side of the equation, while the left-hand side follows from the condition on the path difference.

We see that the electron wavelength is required, which is given by the de Broglie equation

$$\lambda = \frac{h}{p}\,,$$

where h is Planck constant and p is the momentum of the particle. The wavelength for macroscopic objects is negligible, while for microscopic particles, it is comparable to, for example, the inter-atomic distance and interference effects such as diffraction can occur. Our next task is to calculate the electron's momentum.

In classical physics, we can determine the momentum simply from the kinetic energy. The electrons have gained kinetic energy $E_\mathrm{k} = Ue = 90.0\,\mathrm{keV}$ by accelerating in the electric potential $U = 90.0\,\mathrm{kV}$. However, this value is already comparable to their rest mass $E_0 \doteq 511\,\mathrm{keV}$, which is the condition for using special relativity. We can calculate the momentum from the particle's rest energy and its total energy according to the relation

$$E^2 = (E_0 + Ue)^2 = E_0^2 + p^2c^2\,,$$

where $E = E_0 + Ue$ is the particle's total energy. The momentum is therefore

$$p = \frac{1}{c^2}\sqrt{(E_0 + Ue)^2 - E_0^2} = \frac{1}{c}\sqrt{2E_0Ue + U^2e^2}\,.$$

As we now have everything to calculate the angular deviation from the initial direction, let's plug the results into the first relation

$$\sin\alpha = \frac{\lambda}{d} = \frac{h}{pd} = \frac{hc}{d\sqrt{2E_0Ue + U^2e^2}}\,,$$

from which we obtain

$$\alpha \doteq 0.551^\circ \doteq 9.61\,\mathrm{mrad}\,.$$

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Problem FC ... another race of point particles

Lego is preparing for a competition of theoretical physicists competing against each other in a point mass race. The point particles circle a track that goes around a rod of length L lying on the ground. Lego has made its point mass so that its highest acceleration is a and its speed relative to the rod is always the same magnitude. Advise Lego how to choose the magnitude of this velocity so that his point mass circles the bar in the shortest time.

Lego is still too inept to race anything real.

We can assume that the track consists of two identical straight sections and two identical curves. Therefore, it suffices to analyze the total time for one straight section and one curve.

For a speed v, the mass point must move along a circular path (with radius R), for which the following holds

$$a = \frac{v^2}{R} \quad \Rightarrow \quad R = \frac{v^2}{a} .$$

Lego's mass point will move on a semicircular curve with a length of $o/2 = \pi R$, which it will undergo in

$$t_1 = \frac{o/2}{v} = \frac{\pi R}{v} = \frac{\pi v}{a} .$$

If the mass point times the curve correctly, it will not only pass through the point where the rod ends but will also be oriented perpendicularly to the rod at that moment. Then, the straight section of the track will have a length of $L - 2R$, since there is a semicircle of radius R on both sides (here, we tacitly assume that $L > 2R$, though we will later revisit the possibility that this may not hold). The time required for Lego's point to go across the straight section is then

$$t_2 = \frac{L - 2R}{v} = \frac{L - 2\frac{v^2}{a}}{v} = \frac{L}{v} - \frac{2v}{a} .$$

We want to minimize the total time

$$T = t_1 + t_2 = \frac{\pi v}{a} + \frac{L}{v} - \frac{2v}{a} = \frac{L}{v} + (\pi - 2)\frac{v}{a} ,$$

where the parameter we are minimizing over is v. To find the optimal value, we differentiate T with respect to v and set the result equal to zero 0 to obtain the optimal velocity

$$0 = \frac{\mathrm{d}T}{\mathrm{d}v} = -\frac{L}{v^2} + \frac{\pi - 2}{a} ,$$

$$\sqrt{\frac{La}{\pi - 2}} = v .$$

Let us verify our assumption that $L > 2R$, which ensures that the "straight section" has a non-negative length, meaning that if the so-called "optimal velocity" does not satisfy this assumption, it would indicate an incorrect mathematical approach. Evaluating, we find

$$R_{\mathrm{opt}} = \frac{v_{\mathrm{opt}}^2}{a} = \frac{L}{\pi - 2} > \frac{L}{2} .$$

Wait a minute– this actually needed verification. At first, I was surprised too!

The threshold velocity for which $2R = L$ is

$$L = 2R = 2\frac{v^2}{a} \quad \Rightarrow \quad v_0 = \sqrt{\frac{La}{2}}\,,$$

which matches the condition where the "straight section" has zero length. The method we used to compute the time is only valid for velocities $v < v_0$.

For velocities $v > v_0$, the point will simply move in a circular path with the radius we calculated. Since the diameter of this circle is greater than or equal to L, the rod fits inside, meaning the point can indeed orbit around it. In this case, the time for half an orbit is $T = t_1$. Differentiating this time, we get

$$0 = \frac{\mathrm{d}t_1}{\mathrm{d}v} = \frac{\pi}{a}\,,$$

which has no solution. So, the time for $v > v_0$ has a global minimum on the boundary of the interval we are examining. Since its derivative is positive, the minimum occurs at the lower boundary of this interval, i.e., at $v = v_0$. Similarly, $T = t_1 + t_2$ has a negative derivative for all $v < v_0$, meaning the minimum is achieved at

$$v = v_0 = \sqrt{\frac{La}{2}}\,.$$

This simply corresponds to the situation where the point moves in a circle such that the rod is its diameter.

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Problem FD ... oscillating floating plate

Lego saw a plate floating on the surface of water. The plate has thickness $c = 1.0\,\mathrm{cm}$, length $a = 100\,\mathrm{cm}$, width $b = 10\,\mathrm{cm}$ and density $\rho_\mathrm{d} = \rho_\mathrm{v}/2$, where ρ_v is the density of water. So he figured he would use it (how else) for small amplitude oscillations. This time, however, he did not simply push it down, but instead made it oscillate by rotating it a little around its longest axis (which has length a). What will be the period of these small oscillations?

Lego edited his old physics problem so no one could complain that he was stealing.

Let us denote the angle by which Lego rotated the plate as $\mathrm{d}\varphi$. If this angle is small enough, we can ignore the movement of the parts of the plate that remain underwater throughout the rotation and focus only on the sections that emerge or submerge.

We are interested in the torque opposing this rotation. It arises because, on the side that moves downward, a larger portion of the plate becomes submerged, increasing the buoyant force. Conversely, on the opposite side, part of the plate emerges, reducing the buoyant force. Since the plate was initially in equilibrium (and its weight remains unchanged by the rotation), the change in buoyant force at any given point is equal to the net force at that point. As we mentioned, we are interested in the resultant torque, so for each distance from the axis, we calculate the force, multiply it by the distance, and integrate this over all distances.

Thus, we will integrate over thin rectangular strips at a distance x from the axis. Each strip has a width $\mathrm{d}x$ and a fixed length a. For small angles $\mathrm{d}\varphi$, the vertical displacement of the plate at the location of a given strip changes by $x\,\mathrm{d}\varphi$. Thus, the volume of the submerged part will change by

$$\mathrm{d}V = x\,\mathrm{d}\varphi a\,\mathrm{d}x\,.$$

To obtain the elemental force, we multiply by $g\rho_\mathrm{v}$. To obtain the elemental torque, we further multiply by x, giving us

$$\mathrm{d}M = x\,\mathrm{d}F = xg\rho_\mathrm{v}\,\mathrm{d}V = g\rho_\mathrm{v}ax^2\,\mathrm{d}\varphi\,\mathrm{d}x\,,$$

which we integrate from $-b/2$ to $b/2$

$$M = \int_{-\frac{b}{2}}^{\frac{b}{2}} g\rho_\mathrm{v}ax^2\,\mathrm{d}\varphi\,\mathrm{d}x = g\rho_\mathrm{v}a\,\mathrm{d}\varphi\frac{1}{3}\left[x^3\right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{1}{3}g\rho_\mathrm{v}a\frac{b^3}{4}\,\mathrm{d}\varphi\,.$$

The plate is restored to its equilibrium position by a torque, which gives us an angular stiffness of $k_\varphi = (1/12)g\rho_\mathrm{v}ab^3$ times the angle of displacement.

Next, we must determine the plate's angular inertia, i.e., its moment of inertia about the given axis. While this could be computed directly by integration, we can instead use the known result for a homogeneous rectangle with sides b and c, viewed along the axis. Since the plate has a mass of $m = abc\rho_\mathrm{d}$, we can use this to find the moment of inertia

$$J = \frac{1}{12}abc\rho_\mathrm{d}(b^2 + c^2)\,.$$

Thus, the angular acceleration $\varepsilon = M/J$ will be in the opposite direction to the displacement, so the equation of motion will be

$$\mathrm{d}\ddot{\varphi} + \frac{k_\varphi}{J}\,\mathrm{d}\varphi = 0\,,$$

which is the equation of the linear harmonic oscillator. We skip the lecture about its solving and simply plug in the formula for the period of small oscillations

$$T = 2\pi\sqrt{\frac{J}{k_\varphi}} = 2\pi\sqrt{\frac{\frac{1}{12}abc\rho_\mathrm{d}(b^2 + c^2)}{\frac{1}{12}g\rho_\mathrm{v}ab^3}} = 2\pi\sqrt{\frac{c(b^2 + c^2)}{2gb^2}} \doteq 0.14\,\mathrm{s}\,.$$

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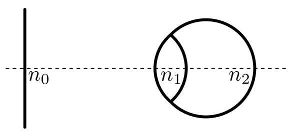
Problem FE ... mobile mirror



Teri was looking at her mobile phone screen from a distance of 20.0 cm. *Suddenly, her reflection on the screen caught her attention and she focused on it. How many times must she have increased the radius of curvature of the lens in her eye if we assume that this lens is symmetrical and thin? The radius of the eyeball is* 1.20 cm, *the refractive index of the air is* $n_0 = 1.00$, *of the lens* $n_1 = 1.42$ *and of the vitreous humor* $n_2 = 1.34$.

Teri was posting stories on FYKOS's instagram.

In this problem we use the following sign convention. The object and image distances are negative to the left of the interface (or positive to the right of the interface), and the radius of curvature of the lens is positive when the center of curvature lies to the right of the interface (or negative when it lies to the left). The distance of the object is therefore $a_1 = -20\,\mathrm{cm}$. The ray goes from left to right, and therefore the centre of curvature of the first interface is positive and that of the second is negative.

The lens is enveloped by air on one side and by vitreous humor on the other. Therefore, for each lens interface, we use the equation for imaging through a spherical interface

$$n_0\left(\frac{1}{a}-\frac{1}{R}\right)=n_1\left(\frac{1}{a'}-\frac{1}{R}\right),$$

where n_0 is the refractive index of the first medium, n_1 is the refractive index of the other medium, a is the object distance, a' is the image distance, and R is is the radius of curvature of the interface (lens).

First, we calculate where the screen is imaged. For the first interface, where by the sign convention the radius of curvature of the lens is positive, we get the equation

$$n_0\left(\frac{1}{a_1}-\frac{1}{R_1}\right)=n_1\left(\frac{1}{s'}-\frac{1}{R_1}\right),$$

where n_0 is the refractive index of air, n_1 is the refractive index of the lens, a_1 is the distance of the object (here the distance of the screen), s' is the distance of the image and R_1 is the radius of curvature of the lens. The inverse of the image distance is therefore

$$\frac{1}{s'}=\frac{n_0}{n_1}\left(\frac{1}{a_1}-\frac{1}{R_1}\right)+\frac{1}{R_1}.$$

The distance of this image is now determined as the distance of the object to be viewed through the second lens interface $s'=s$. Again, we use the equation for imaging through a spherical interface

$$n_1\left(\frac{1}{s}+\frac{1}{R_1}\right)=n_2\left(\frac{1}{a_1'}+\frac{1}{R_1}\right),$$

where the sign in front of $1/R_1$ has changed because R_1 is negative due to sign convention (however, due to the lens symmetry, the absolute value is the same as for the first interface). Furthermore, in the equation a_1' is the distance of the object, n_2 is the refractive index of the vitreous humor, and for $1/s$ we substitute the result of the previous equation. If the image is supposed to be focused, it must appear on the retina, i.e. $a_1'=2r$, where r is the radius of the eye.

Overall, we express the radius of curvature from the previous equation

$$R_1=\frac{2a_1r(2n_1-n_0-n_2)}{2n_0r-n_2a_1}.$$

Looking at the reflection, Teri focused on it. This reflection is at the same distance from the screen as she is, so we get a new distance $a_2=2a_1$ for the object, and we can express the new radius of curvature R_2 using the previous equation as

$$R_2=\frac{4a_1r(2n_1-n_0-n_2)}{2n_0r-2n_2a_1}.$$

When we express the ratio of R_1 and R_2, we get the resulting change in radius of curvature

$$\frac{R_2}{R_1} = \frac{2n_0 r - n_2 a_1}{n_0 r - n_2 a_1} \doteq 1.04\,.$$

Thus, Teri had to increase the radius of curvature of her lens by 4 %.

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Problem FF ... point mass in a round corner

Consider a plane bounded by a boundary in the form of a circle with radius $R = 15\,\mathrm{m}$. A point mass lies inside, close to the boundary. It is suddenly sent with velocity $v_0 = 30\,\mathrm{m\cdot s^{-1}}$ tangent to the boundary. The friction between the pad and the mass point is given by the friction coefficient $f = 0.60$, as is the friction between it and the mantel. How long does it take for the point to stop? *Jarda was inspired by his favourite assignement.*

The frictional force is defined as

$$F_{\mathrm{T}} = -fF_{\mathrm{N}}\,,$$

where F_{N} is the normal force on the base or the boundary, and the minus sign indicates that the frictional force acts against the direction of motion of the point mass. For friction between the point mass and the base, the friction force equals

$$F_{\mathrm{T}} = -fmg\,.$$

For the friction between the point mass and the boundary, we have

$$F_{\mathrm{TO}} = -fm\frac{v^2}{R}\,.$$

The coefficient v^2/R represents the centrifugal acceleration acting on the mass point

$$a = \omega^2 R = \frac{v^2}{R}\,.$$

The equation of motion for a mass point, where we consider velocity v instead of position x, can be written as

$$m\dot{v} = -fmg - fm\frac{v^2}{R}\,.$$

As it is a linear differential equation with separated variables, we can solve it via integration with substitution $u = v/\sqrt{gR}$

$$\frac{1}{fg}\dot{v} = -1 - \frac{v^2}{gR}\,,$$

$$-\frac{1}{fg}\frac{\dot{v}}{1 + \frac{v^2}{gR}} = 1\,,$$

$$-\frac{1}{fg}\int \frac{1}{1 + \left(\frac{v}{gR}\right)^2}\,\mathrm{d}v = t\,,$$

$$-\frac{1}{f}\sqrt{\frac{R}{g}}\int \frac{1}{1+(u)^2}\,\mathrm{d}u = t\,,$$
$$-\frac{1}{f}\sqrt{\frac{R}{g}}\arctan\frac{v}{\sqrt{gR}} = t + t_0\,,$$
$$v = -\sqrt{gR}\tan\left(f\sqrt{\frac{g}{R}}\,(t+t_0)\right).$$

The integration constant t_0 can be expressed from the initial condition $v(0) = v_0$ as

$$t_0 = -\frac{1}{f}\sqrt{\frac{R}{g}}\arctan\left(\frac{v_0}{\sqrt{gR}}\right).$$

The overall solution for the velocity is then

$$v(t) = -\sqrt{gR}\tan\left(f\sqrt{\frac{g}{R}}t - \arctan\left(\frac{v_0}{\sqrt{gR}}\right)\right).$$

Now, we still need to determine when the velocity v will be zero. Luckily for us, the tangent function takes the value 0 trivially at 0, which corresponds to the situation when

$$t = -t_0 = 2.4\,\mathrm{s}\,,$$

where t_0 was expressed earlier in the search for a solution for velocity.

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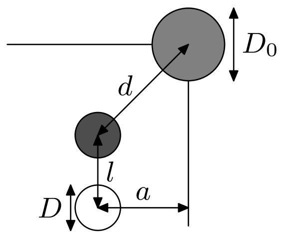
Problem FG ... billiards



Petr is playing billiards. Only the black ball remains, and it is located at a distance $d = 30\,\mathrm{cm}$ from the center of a pocket with a diameter $D_0 = 11.4\,\mathrm{cm}$, The black ball is also $a = 10\,\mathrm{cm}$ from one of the table rails. The white ball is $l = 1.0\,\mathrm{m}$ away from the black ball and is at the same distance a from the same rail as the black ball. What is the minimum kinetic energy Petr has to transfer to the white ball with the pool cue so that the black ball falls into the pocket? Both balls decelerate uniformly with a deceleration of $\alpha = 30\,\mathrm{cm{\cdot}s^{-2}}$, and Petr aims the black ball at the center of the pocket. Assume both balls have the same mass $m = 160\,\mathrm{g}$ and diameter $D = 5.7\,\mathrm{cm}$, and their collision is perfectly elastic. Neglect ball rotation. All provided distances are measured from the centers of the balls. *Peter used to play billiards with his friends.*

The ball will fall into the pocket when its center passes beyond the pocket's edge. Since the ball also decelerates uniformly with the rate of deceleration α, we can write the conditions for the critical velocity of the black ball as.

$$v_\mathrm{m} - \alpha t = 0\,,$$
$$v_\mathrm{m} t - \frac{1}{2}\alpha t^2 = d - \frac{D_0}{2}\,,$$

where t is the time it takes for the black ball to reach the edge of the pocket. From the first equation, we can solve for t, and then substitute it into the second equation to find the magnitude of the critical velocity v_{m}

$$v_{\mathrm{m}} = \sqrt{\alpha(2d - D_0)} \doteq 38\,\mathrm{cm{\cdot}s^{-1}}\,.$$

When the white ball strikes the black ball, it transfers velocity to it according to conservation of momentum principle. However, we only consider the component of velocity along the line connecting the centers of the balls at the moment of collision. Notice that the white ball transfers the entire velocity component to the black ball, because both balls have the same mass m and the black ball is initially at rest. This can be written as

$$v_{\mathrm{m}} = v\cos\gamma\,,$$

where v is the velocity of the white ball before the collision (or the velocity of the black ball after the collision), and γ is the angle between the velocity vector of the white ball and the line connecting the centers of the balls at the moment of collision. Now, let us consider the geometry of the problem, referring to the diagram provided. For the angle θ between the horizontal direction and the line connecting the centers of the black ball and the pocket, we have

$$\cos\theta = \frac{a}{d}\,,$$
$$\theta = \arccos\frac{a}{d} \doteq 70.5^\circ\,.$$

Since d is the shortest distance between the black ball and the pocket, we want to shoot the black ball under this angle. Therefore, we want the white ball to strike the black ball such that the angle between the line connecting the centers of the balls and the horizontal direction during the collision is equal to θ. Let us denote by φ the angle at which we need to shoot the white ball. By analyzing geometry, we get

$$\tan\varphi = \frac{D\cos\theta}{l - D\sin\theta}\,,$$
$$\varphi = \arctan\frac{D\cos\theta}{l - D\sin\theta} \doteq 1.15^\circ\,.$$

For γ, we have

$$\gamma = \frac{\pi}{2} - \theta + \varphi \doteq 20.6^\circ\,.$$

Now, let us we determine the minimum initial velocity v_0. Let L be the distance traveled by the white ball before the collision. We can calculate

$$L = \frac{D\cos\theta}{\sin\varphi} \doteq 95\,\mathrm{cm}\,.$$

For the velocity v_0, the following must hold

$$v_0 - \alpha T = v\,,$$
$$v_0 T - \frac{1}{2}\alpha T^2 = L\,,$$

where T is the time it takes for the white ball to travel the distance L. Solving for T from the first equation and substituting it into the second equation, we find v_0 as

$$v_0 = \sqrt{2\alpha \frac{D\cos\theta}{\sin\varphi} + \frac{\alpha\,(2d - D_0)}{\cos^2\gamma}} \doteq 86\,\mathrm{cm{\cdot}s^{-1}}\,.$$

Finally, we substitute into the well-known formula for kinetic energy

$$E_\mathrm{k} = \frac{1}{2} m v_0^2 = m\alpha \left(\frac{D\cos\theta}{\sin\varphi} + \frac{\left(d - \frac{D_0}{2}\right)}{\cos^2\gamma} \right) \doteq 59\,\mathrm{mJ}\,.$$

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Problem FH . . . pencil on the edge

A pencil of length l with negligible thickness is placed on the right edge of the table so that $3/5$ of its length are on the table and the remaining $2/5$ are hanging in the air. What is the greatest distance $d_{\max}$ from the table at which we can strike the pencil from below with an upward impulse of force such that the pencil is immediately lifted off the table in its entirety (and thus not merely tipped over its left end)? Provide the result as the ratio $d_{\max}/l$.

Kuba was playing with a pencil during the lecture.

Let us denote the magnitude of the impulse of the force by I, the distance of the impulse of the force from the center of gravity of the pencil by x and the mass of the pencil by m. Note that to actually apply the impulse in the air and not on the table, the following must hold

$$x > \frac{l}{10}\,.$$

The impulse of the force gives the pencil a momentum p and a moment L with respect to the axis of rotation passing through its centre of gravity. If the pencil is to be lifted off the table immediately (and thus receives no additional impulse of force from the table), we get from the impulse theorems

$$\Delta p = F\Delta t = I \quad \Rightarrow \quad p = I\,,$$
$$\Delta L = xF\Delta t = xI \quad \Rightarrow \quad L = Ix\,.$$

Now let us examine the vertical displacement of the left end of the pencil $\mathrm{d}y_0$ at the first infinitesimal time point $\mathrm{d}t$. For the vertical y-coordinate of the centre of gravity y we have

$$p = m\frac{\mathrm{d}y}{\mathrm{d}t} \quad \Rightarrow \quad \mathrm{d}y = \frac{p\,\mathrm{d}t}{m} = \frac{I\,\mathrm{d}t}{m}\,.$$

At the same time, the pencil rotates around its centre of gravity by $\mathrm{d}\theta$, which satisfies

$$L = J\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{12} m l^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} \quad \Rightarrow \quad \mathrm{d}\theta = \frac{12L\,\mathrm{d}t}{ml^2} = \frac{12Ix\,\mathrm{d}t}{ml^2}\,,$$

where we have used the relation for the moment of inertia of a thin rod with respect to the axis through its centre of gravity $J = (1/12)ml^2$.

From the geometry, we see that the vertical displacement of the left end of the pencil $\mathrm{d}h$ caused by rotation by $\mathrm{d}\theta$ is

$$-\tan \mathrm{d}\theta = -\mathrm{d}\theta = \frac{2\,\mathrm{d}h}{l} \quad \Rightarrow \quad \mathrm{d}h = -\frac{l\,\mathrm{d}\theta}{2} = -\frac{6Ix\,\mathrm{d}t}{ml}$$

and thus we get

$$\mathrm{d}y_0 = \mathrm{d}y + \mathrm{d}h = \frac{I\,\mathrm{d}t}{m}\left(1 - \frac{6x}{l}\right).$$

The pencil will detach from the table when $\mathrm{d}y_0 > 0$, which gives the condition

$$\left(1 - \frac{6x}{l}\right) > 0 \iff x < \frac{l}{6}.$$

However, the task was to find out the maximum distance from the edge of the table

$$d_{\max} = x_{\max} - \frac{l}{10} = \frac{l}{15}.$$

Thus, we obtain the final result

$$\frac{d_{\max}}{l} = \frac{1}{15}.$$

Let us also comment on the fact that the solution of the problem is not affected by the gravitational acceleration. The condition $\mathrm{d}y_0 > 0$ means in terms of derivatives

$$\dot{y}_0(0) > 0. \tag{1}$$

Thus, the function $y_0(t)$ must be increasing (which is exactly what we want). However, the gravitational acceleration will always be proportional to the term $\ddot{y}_0$ and therefore will not eventually play a role in the (1) condition (thus affecting its convexity/concavity, but not its monotonicity).

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Problem GA ... let's not neglect Saturn

What is the ratio of the gravitational force acting on a body of mass $m = 10.0\,\mathrm{kg}$ by Saturn to that acting on it by Saturn's rings? Assume that the body is exactly on the axis of the rings at a distance $D = 100\,000\,\mathrm{km}$ from the centre of the planet. Consider further that the rings are a homogeneous disk with a constant surface density $\rho = 315\,\mathrm{kg{\cdot}m^{-2}}$ with the inner radius $R_0 = 67\,300\,\mathrm{km}$ and the outer radius $R_1 = 140\,000\,\mathrm{km}$ centered at the center of Saturn. Saturn has a mass of $M_\mathrm{S} = 5.68 \cdot 10^{26}\,\mathrm{kg}$. *Pepa decided to get out of his comfort zone.*

First, we calculate the gravitational force exerted by Saturn on a body of mass m

$$F_\mathrm{S} = G\frac{M_\mathrm{S} m}{D^2}.$$

Let's also determine the force from the rings. Consider a thin ring with an area of $\mathrm{d}S = 2\pi r\,\mathrm{d}r$, located at a distance r from the planet's center. Since every point on the ring is equidistant from the body, the forces acting toward the planet add up, while those in the plane perpendicular to the line connecting the body and the planet's center cancel out. The force exerted by this thin ring on the body is given by

$$dF_{\mathrm{p}} = G\frac{m\rho 2\pi r\,\mathrm{d}r}{r^2+D^2}\frac{D}{\sqrt{r^2+D^2}}\,,$$

where $r^2 + D^2$ represents the square of the ring's distance from the body, and $D/\sqrt{r^2+D^2}$ accounts for the component of the force directed toward the planet's center. The resulting force is then obtained by integrating from R_0 to R_1 (this integral can be solved by substituting t for $r^2 + D^2$).

$$F_{\mathrm{p}} = -2mD\rho\pi G\left(\frac{1}{\sqrt{R_1^2+D^2}} - \frac{1}{\sqrt{R_0^2+D^2}}\right)\,.$$

Then, we substitute in the ratio

$$\frac{F_{\mathrm{S}}}{F_{\mathrm{p}}} = \frac{M_{\mathrm{S}}\sqrt{(R_0^2+D^2)(R_1^2+D^2)}}{2D^3\rho\pi\left(\sqrt{R_1^2+D^2}-\sqrt{R_0^2+D^2}\right)} \doteq 1.16\cdot 10^8\,.$$

Thus, the force exerted on the body by Saturn is more than 10^8 times stronger than the force from its rings.

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Problem GB ... electrostatic pyramid

Imagine a square with sides of length $a = 0.5\,\mathrm{m}$. Positive point charges of size $Q = 15\,\mathrm{\mu C}$ are placed in the square's vertices. We construct a fixed axis perpendicular to the square's surface passing through the center of the square. We thread a bead of mass $m = 80\,\mathrm{g}$ charged with a positive charge $q = 2.0\,\mathrm{\mu C}$ to the axis. What would be the period of small oscillations of the bead? *Peter remembered Egypt.*

Let us denote the height of the bead along the axis as z. For the gravitational force, we have

$$F_{\mathrm{T}} = mg\,.$$

The electrostatic force can be determined using the formula

$$F_{\mathrm{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R^2}\,,$$

where ε_0 is the vacuum permittivity, and R the distance of the bead from the charge Q. Using geometry, we can express this distance using the Pythagorean theorem as

$$R = \sqrt{\frac{a^2}{2}+z^2}\,.$$

It is helpful to realize that the x and y components of the forces acting on the charge q from the opposite charges Q cancel each other out (and the bead's motion is restricted to the z-direction). Thus, we only need to consider the z component of the force F_{E}, i.e., its projection onto the z-axis

$$F_{\mathrm{E},z} = \frac{Qq\cos\theta}{2\pi\varepsilon_0}\frac{1}{a^2+2z^2}\,,$$

where θ is the angle between the axis and the line connecting the charges Q, q. With the help of geometry, we can express

$$\cos\theta = \frac{\sqrt{2}z}{\sqrt{a^2+2z^2}}\,,$$

and then obtain

$$F_{\mathrm{E},z} = \frac{\sqrt{2}Qq}{2\pi\varepsilon_0}\frac{z}{\left(a^2+2z^2\right)^{3/2}}\,.$$

Since there are four charges, the total force is the sum of four times the force $F_{\mathrm{E},z}$ and the gravitational force F_{T}

$$F = \frac{2\sqrt{2}Qq}{\pi\varepsilon_0}\frac{z}{\left(a^2+2z^2\right)^{3/2}} - mg\,.$$

Now, we calculate the period of small oscillations (with the help of Taylor expansion). The equation of motion for the bead is

$$m\ddot{z} - \frac{2\sqrt{2}Qq}{\pi\varepsilon_0}\frac{z}{\left(a^2+2z^2\right)^{3/2}} + mg = 0\,.$$

We can rearrange this equation by defining the stable position of the bead $F(z_0) = 0$ and introducing a new coordinate (displacement) $\xi = z - z_0$. To find z_0 we would have to solve a cubic equation, but for now, we will find its value numerically using a calculator. We will come back to it later. Now, we can linearize the force dependence $F(\xi)$ by assuming $\xi \ll z_0$.

$$\frac{z_0+\xi}{\left(a^2+2(z_0+\xi)^2\right)^{3/2}} \approx \frac{z_0}{\left(a^2+2z_0^2\right)^{3/2}} + \frac{\xi}{\left(a^2+2z_0^2\right)^{3/2}} - \frac{6z_0^2\xi}{\left(a^2+2z_0^2\right)^{5/2}} =$$
$$= \frac{\pi\varepsilon_0 mg}{2\sqrt{2}Qq} - \frac{4z_0^2-a^2}{\left(a^2+2z_0^2\right)^{5/2}}\xi \quad \Rightarrow \quad \ddot{\xi} + \frac{2\sqrt{2}Qq}{\pi m\varepsilon_0}\frac{4z_0^2-a^2}{\left(a^2+2z_0^2\right)^{5/2}}\xi = 0\,.$$

Before solving this differential equation (for example, via variation of parameters), we recall that the general form of the equation for a harmonic oscillator is

$$\ddot{y} + \omega^2 y = 0\,,$$

where ω is the angular frequency of oscillation. So, if the factor in front of ξ is positive (i.e., if $z_0 > a/2$), the frequency will be real, and we can compute it as

$$\omega = \sqrt{\frac{2\sqrt{2}Qq}{\pi m\varepsilon_0}\frac{4z_0^2-a^2}{\left(a^2+2z_0^2\right)^{5/2}}}\,.$$

We can calculate the period of oscillation using the relation $\omega = 2\pi/T$

$$T = \sqrt{\frac{\sqrt{2}\pi^3 m\varepsilon_0}{Qq}\frac{\left(2z_0^2+a^2\right)^{5/2}}{4z_0^2-a^2}}\,.$$

To find the numerical value of the result, we still need to compute the value of z_0. We can do this using an iterative method, which is often used to solve analytically unsolvable equations. It involves isolating the variable we seek on one side of the equation. The other side will contain an expression that still includes this variable, so we substitute it with an initial estimate. By computing the value of the expression, we get a better estimate for the sought value, which we can then substitute again. This process is iteratively repeated on the calculator until the result no longer changes (the sequence converges to the solution). In our case, we have two ways to express the sought z_0 from the equation $F(z_0) = 0$, each of which converges to a different solution.

$$\frac{2\sqrt{2}Qq}{\pi\varepsilon_0}\frac{z_0}{\left(a^2+2z_0^2\right)^{3/2}} - mg = 0 \Rightarrow$$

$$\Rightarrow \begin{cases} z_{0,n} = \frac{\pi\varepsilon_0 mg}{2\sqrt{2}Qq}\left(a^2 + 2z_{0,n-1}^2\right)^{3/2} & \to z_{0,\infty} \doteq 0.033\,\mathrm{m} \\ z_{0,n} = \sqrt{\left(\frac{Qq\, z_{0,n-1}}{\pi\varepsilon_0 mg}\right)^{2/3} - \frac{a^2}{2}} & \to z_{0,\infty} \doteq 1.087\,\mathrm{m} \end{cases}$$

As we can see, only the second solution satisfies the condition $z_0 > a/2$, so we have $z_0 \doteq \doteq 1.09\,\mathrm{m}$. We can now substitute this into the derived expression for the period of the bead and get $T \doteq 1.6\,\mathrm{s}$.

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Problem GC ... heavenly aquarium

There is an aquarium in heaven with a height of $h = 0.50\,\mathrm{m}$ filled with holy water, the refractive index of which decreases linearly with height so that at the bottom it has the value $n_0 = 1.6$ and at the surface $n_1 = 1.3$. Saint Peter directed a laser beam into the aquarium from below with an angle of incidence of 45°. Determine the horizontal distance from the point at which the beam entered the aquarium at the bottom to the point at which it exited at the surface. Consider the refractive index of air $n_{\mathrm{vz}} = 1$ (exactly) and that the walls of the aquarium are negligibly thin.

Hint:

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \operatorname{argcosh} x, \quad x \in \langle 1, \infty) \,.$$

The same thought enlightened both Kuba and Petr.

Let us introduce a coordinate system with a horizontal axis x, a vertical axis y, and the origin $(0, 0)$ at the point where the ray enters the aquarium. Let $y(x)$ denote the actual trajectory of the ray in the aquarium. Before we start solving the problem itself, let us simplify the situation a bit. Let us divide the aquarium into several equidistant layers, each with a constant refractive index n_k. In each layer, the ray will then propagate along a straight line. If we denote by α_k the angle that the k-th such line makes with the vertical, Snell's law states that the quantity $n_k \sin\alpha_k$ is constant for all k. It is easy to realize that we can generalize this idea for infinitely many layers, i.e. for a continuous change in the refractive index. For a ray entering the aquarium from the air, in this limit we get

$$n(y)\sin\alpha(y) = n_{\mathrm{vz}}\,\sin 45^\circ = \frac{1}{\sqrt{2}} =: a_0\,. \tag{2}$$

The refractive index n as a function of height y can be expressed as

$$n(y) = n_0 + \frac{n_1 - n_0}{h} y =: n_0 + ky\,.$$

For the angle α from the geometry of a right triangle and from the properties of the derivative we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cot\alpha = \frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha} \quad \Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{1+(y')^2}}\,.$$

The equation (2) can thus be modified to the form

$$\frac{n_0 + ky}{\sqrt{1+(y')^2}} = a_0\,.$$

This is an ordinary differential equation that can be converted into a separated form by algebraic manipulations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{\frac{(n_0+ky)^2}{a_0^2} - 1}\,,$$

$$\frac{\mathrm{d}y}{\sqrt{\frac{(n_0+ky)^2}{a_0^2} - 1}} = \pm\,\mathrm{d}x\,. \tag{3}$$

In the following, we will consider only the variant with a plus, we will return to the second case later. We can now integrate the equation (3). On its left side we get

$$\int \frac{\mathrm{d}y}{\sqrt{\frac{(n_0+ky)^2}{a_0^2} - 1}} = \frac{a_0}{k}\int\frac{\mathrm{d}u}{\sqrt{u^2-1}} = \frac{a_0}{k}\operatorname{argcosh} u\ ,$$

where $u = (n_0 + ky)/a_0$. We can easily see that $u > 1$, and $\operatorname{argcosh} u$ is thus indeed defined.

We integrate the right side trivially and get

$$\frac{a_0}{k}\operatorname{argcosh}\frac{n_0+ky}{a_0} = x - \xi, \tag{4}$$

where ξ is the integration constant. If we now express the coordinate y from the equation (4), we obtain

$$y = \frac{1}{k}\left(a_0\cosh\frac{k\,(x-\xi)}{a_0} - n_0\right)\,. \tag{5}$$

Since the hyperbolic cosine is an even function, we see that the sign in the equation (3) does not matter. It remains to determine the parameter ξ. We express it simply from the equation (**??**) using the boundary condition $y(0) = 0$

$$\xi = -\frac{a_0}{k}\operatorname{argcosh}\frac{n_0}{a_0}\,.$$

The resulting curve has the shape of a *catenary*, which in this case forms a concave arc. This is exactly the shape of freely hanging chains. We will not delve into how surprising this result is but simply note that both problems can be elegantly formulated using the *calculus of variations*, which leads to the same variational problem in both cases.

We find the horizontal distance of the second intersection point by substituting $y = h$ into the resulting relation (5). As we would expect, we get two solutions

$$x = \xi \pm \frac{a_0}{k} \operatorname{argcosh} \frac{n_0 + kh}{a_0} ,$$

where the larger solution corresponds to the unphysical case where the beam, after emerging from the aquarium at the surface, turns around and re-enters. We can easily see that the correct choice is the plus sign and the distance we are looking for is

$$\begin{aligned} \Delta x &= \frac{a_0}{k} \operatorname{argcosh} \frac{n_0 + kh}{a_0} - \frac{a_0}{k} \operatorname{argcosh} \frac{n_0}{a_0} , \\ &= \frac{a_0}{k} \left(\operatorname{argcosh} \frac{n_1}{a_0} - \operatorname{argcosh} \frac{n_0}{a_0} \right) \doteq 0.28\,\mathrm{m} . \end{aligned}$$

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Problem GD ... strange equilibrium

Consider a non-conductive cylinder with a cross-sectional area $S = 10\,\mathrm{cm^2}$. The cylinder is sealed on both ends with metal pistons, trapping some amount of ideal gas inside at a temperature of $T = 300\,\mathrm{K}$. The external pressure is $p_\mathrm{a} = 101\,\mathrm{kPa}$, and the distance between the pistons stabilizes on $d_0 = 1.0\,\mathrm{mm}$. Then, the pistons are connected to a battery, which applies a voltage of $U = 250\,\mathrm{V}$ between the pistons. By how much will the pistons move closer due to the applied voltage? Write a negative sign if they come further apart. Assume that $d^2 \ll S$, the temperature inside the cylinder remains constant at T due to the heat exchange with the surroundings, and the relative permittivity $\varepsilon_\mathrm{r} = 1$. *Lego*
has wanted a problem like this problem for a long time but it was a pain to make it work.

When there is a potential difference between the pistons, it means that there are opposite charge densities on them, and opposite charges attract each other. On the other hand, the gas enclosed between the pistons will push them apart, but the surrounding atmosphere will push them together. Therefore, the pistons will reach an equilibrium position where these two forces cancel each other out. Let us calculate the magnitudes of these forces as a function of the distance between the pistons.

We begin with the pressure. We use the ideal gas law in a form of $p = nRT/V$, where n and R are constant, T is given, and V is the volume between the pistons. Since the cylinder has a cross-sectional area S, for a piston distance d, the volume will be $V = Sd$. Thus, the pressure of the gas becomes $p = nRT/(Sd)$. On the opposite side, the atmospheric pressure p_a pushes on the piston. Therefore, the net force exerted by the air on the piston is:

$$F_\mathrm{p} = S\Delta p = \frac{nRT}{d} - p_\mathrm{a} S ,$$

From this, we can calculate that the number of particles n enclosed between the pistons is $n = p_\mathrm{a} S d_0/(RT)$.

For sufficiently small distances d, the pistons will behave like a capacitor. The capacitance of this capacitor is $C = \varepsilon_0 S/d$, so the charge on the pistons will be $Q = UC = U\varepsilon_0 S/d$, where U is the potential difference between the pistons. This charge will be attracted by the electric field produced by the charge on the opposite piston. For $d^2 \ll S$, this electric field will be the same as the field produced by an infinite plate charged with a surface charge density $\sigma = Q/S$, that is

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{2S\varepsilon_0} \,.$$

Be careful! We must consider the field from the second plate and not the field inside the capacitor (which is twice as strong), because that would imply the plate "acting on itself". Mathematically, the field is discontinuous at the plate. The force exerted on the piston due to this electric field is then:

$$F_Q = EQ = \frac{Q^2}{2S\varepsilon_0} = \frac{U^2\varepsilon_0^2 S^2}{2d^2 S\varepsilon_0} = \frac{U^2\varepsilon_0 S}{2d^2} \,.$$

The magnitudes of the two forces must be equal, from which we can express the distance d for which they cancel each other out.

$$\begin{aligned} F_\mathrm{p} &= F_Q \,, \\ \frac{nRT}{d} - p_\mathrm{a} S &= \frac{U^2\varepsilon_0 S}{2d^2} \,, \\ 0 &= d^2 p_\mathrm{a} S - dnRT + \frac{1}{2}U^2\varepsilon_0 \,, \\ Sd &= \frac{nRT \pm \sqrt{(nRT)^2 - 2S^2 p_\mathrm{a} U^2 \varepsilon_0}}{2p_\mathrm{a} S} \,, \\ d &= \frac{p_\mathrm{a} S d_0 \pm \sqrt{(p_\mathrm{a} S d_0)^2 - 2S^2 p_\mathrm{a} U^2\varepsilon_0}}{2p_\mathrm{a} S} \,, \\ d &= \frac{d_0 \pm \sqrt{d_0^2 - 2U^2\varepsilon_0/p_\mathrm{a}}}{2} \,, \end{aligned}$$

where in the penultimate step, we substituted from the initial condition $p_\mathrm{a} S d_0 = nRT$.

Now, we need to decide which root to consider. For $U = 0$, we should get $d = d_0$, which we obtain exactly for the root with the plus sign, so we choose that one. The other root is unstable – while the forces balance out, a small decrease in d would make the attractive force stronger than the repulsive force, and the plates would approach each other to zero distance (which is physically unrealistic). On the other hand, if d increases slightly, the system will stabilize at the correct root.

Now, we can either directly substitute and subtract from d_0, or notice that $d_0^2 \gg 2U^2\varepsilon_0/p_\mathrm{a}$ and perform a first order Taylor expansion

$$d = \frac{d_0 + d_0\sqrt{1 - \frac{2U^2\varepsilon_0}{p_\mathrm{a} d_0^2}}}{2} = d_0 - \frac{U^2\varepsilon_0}{2p_\mathrm{a} d_0} \,.$$

Thus, the pistons approach by $\Delta d = U^2\varepsilon_0/(2p_\mathrm{a} d_0) \doteq 2.7\,\mathrm{nm}$, which corresponds to a few dozen atoms.

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Problem GE ... a snow globe

Jarda has a decorative hemisphere with a Christmas motif (a snow globe). It has a radius $R = 3.3\,\mathrm{cm}$ and is completely filled with water. Inside the globe, figurines are preparing to skate on a frozen pond, represented by a reflective surface around the axis of symmetry. Elsewhere, the base material of the snow globe is matte. The decoration is placed on a table, and one day, Jarda illuminated it with a point light source positioned along the hemisphere's axis of symmetry at a height of $h = 29\,\mathrm{cm}$ above the table. At what height above the highest point of the hemisphere will the rays reconverge after interacting with the object?

Jarda was inventing promotional items for Výfuk.

If the reflective part is only near the symmetry axis of the snow globe, we will consider only the rays traveling close to the optical axis of the system. Therefore, we can remain in the approximation of classical geometrical optics, where all rays converge at a single point. Another significant step is to realize that the mirror inverts the formed image around the plane of symmetry – the base of the snow globe. However, nothing prevents us from flipping the situation across the base plane and solving the problem in that setup. Thanks to the properties of the mirror, our solution will be the same. Now, however, we are essentially computing the passage of paraxial rays through a glass sphere because the hemisphere of the snow globe is conceptually completed by another hemisphere behind the mirror. Thus, with a simple consideration, we have transformed the problem into a somewhat simpler one.

Still, the problem is more complicated than it may seem at first. On the internet or in textbooks, one can find pre-derived formulas for the focal length of a spherical lens, but we need to derive them ourselves. The challenge is that this is not a thin lens, which we may be familiar with from other problems, but rather a thick lens, where the distance between the spherical interfaces is non-negligible (approximately $2R$). Consequently, not all known relationships apply here. There are different methods for solving problems involving thick lenses, and we will choose the least direct one – computing the ray's path from the source through individual regions and determining the point where it intersects the optical axis again.

Let a be the distance of the source from the center of the sphere, and a' be the distance of the image from the center. According to the problem statement, we will eventually substitute $a = h$, and our answer will be $h' = a' - R$.

Consider a ray emanating from the source at an angle α to the optical axis, intersecting the sphere-air interface at point A, which is at a distance y from the optical axis. For this point, we have

$$y = (a - R)\,\alpha = R\varphi\,,$$

thus defining the angle φ, which the line connecting the sphere's center and point A makes with the optical axis. Recall that in the paraxial approximation, we work with small angles, so we use the approximations $x \approx \sin x \approx \tan x$.

At point A, the ray refracts according to Snell's law. In the air, the ray's direction forms an angle $\beta = \alpha + \varphi$ with the normal to the sphere's surface. Inside, let the angle between the ray and the line connecting the sphere's center and point A be γ. Then

$$\beta = n\gamma\,,$$

which is the Snell's law form in our approximation, assuming the refractive index of air is one. The angle γ thus depends only on the refractive index of water n. The glass maintaining the snow globe's shape is assumed to be very thin, so we neglect it.

The ray continues through the sphere until it reaches the opposite interface, where it refracts again and eventually intersects the optical axis at the already defined distance a'. Let us define the second refraction point analogously to the first as A', located at a distance y' from the optical axis. Define angles γ', β', and α' similarly to the previous interface. Here, we have $\beta' = n\gamma'$.

The problem is now well-defined and we only need to determine the distance a'. Notice that the triangle comprised of points A, A', and the sphere center S is an isosceles triangle with sides of length R. Thus, we have $\gamma = \gamma'$, which also implies $\beta = \beta'$, leading to

$$\alpha + \varphi = \alpha' + \varphi' \quad \Rightarrow \quad \frac{y}{a - R} + \frac{y}{R} = \frac{y'}{a' - R} + \frac{y'}{R}\,.$$

The position of a' should not depend on y or y', so we must find another equation that relates these two variables linearly. We have not yet considered the distance between the two interfaces, so let us describe it using

$$\begin{aligned} y - y' &= 2R\left(\gamma' - \varphi'\right) = 2R\gamma' - 2R\varphi' = 2R\gamma' - 2y'\,, \\ y + y' &= 2R\gamma\,. \end{aligned}$$

Using Snell's law, we successively substitute for γ

$$\begin{aligned} y + y' &= \frac{2R}{n}\beta = \frac{2R}{n}\left(\alpha + \varphi\right) = \frac{2R}{n}\left(\frac{y}{a - R} + \frac{y}{R}\right)\,, \\ y' &= y\frac{2R}{n}\left(\frac{1}{a - R} + \frac{1}{R} - \frac{n}{2R}\right)\,. \end{aligned}$$

Substituting this into our previous equation involving y and y' leaves us with

$$\frac{1}{a - R} + \frac{1}{R} = \left(\frac{1}{a' - R} + \frac{1}{R}\right)\frac{2R}{n}\left(\frac{1}{a - R} + \frac{1}{R} - \frac{n}{2R}\right)\,,$$

from which we now express a'

$$\begin{aligned} \frac{n}{2R}\frac{\frac{1}{a-R} + \frac{1}{R}}{\frac{1}{a-R} + \frac{1}{R} - \frac{n}{2R}} - \frac{1}{R} &= \frac{1}{a' - R}\,, \\ \frac{n}{2R}\left(\frac{1}{1 - \frac{nR(a-R)}{2Ra}} - \frac{2}{n}\right) &= \frac{1}{a' - R}\,, \\ a' &= R\frac{2a - na + nR}{2a\left(n - 1\right) - nR} + R\,, \\ a' &= R\frac{a\left(2 - n\right) + 2a\left(n - 1\right)}{2a\left(n - 1\right) - nR}\,, \\ a' &= \frac{nRa}{2a\left(n - 1\right) - nR} \doteq 8.6\,\mathrm{cm}\,. \end{aligned}$$

To correctly answer the problem, we must subtract the sphere's radius

$$h' = a' - R \doteq 5.3\,\mathrm{cm}\,.$$

The object is displayed at a distance of 5.3 cm above the top of the object. Incidentally, using the Gaussian lens equation, we can express the focal length of the sphere as:

$$\frac{1}{a}+\frac{1}{a'}=\frac{1}{f}\quad\Rightarrow\quad f=\frac{2R}{2\,(n-1)}\,.$$

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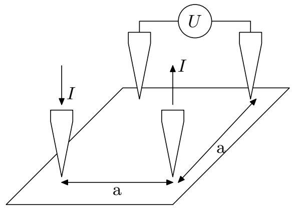
Problem GF ... surface conductivity

Let's consider an infinitely large conductive plane. We connect two electrodes to two points separated by a distance of a, allowing a current I to flow between them. Another pair of electrodes is attached to the surface for voltage measurement. Together with the first two electrodes, they form a square on the surface. The measured voltage between them is U. What is the surface conductivity of the plane? *Jarda simplified Karel's task.*



We must first compute the electric field in the whole plane to find the potential between the corners with voltage contacts. Then, we can obtain the potential by integrating the field between these points. To calculate the field $\mathbf{E}$, we use Ohm's law in differential form

$$\mathbf{j}_\mathrm{s}=\sigma_\mathrm{s}\mathbf{E}\,,$$

where σ_s is the surface conductivity, and $\mathbf{j}_\mathrm{s}$ is the surface current density, which describes the charge flow per unit length per unit time in a given point.

In this problem, the surface current density is easily determinable due to the symmetry of the setup (the infinite extent of the plane, and the principle of superposition). We introduce a 2D coordinate system with a conductor attached at the origin, injecting a current I into the plane. The current must spread isotropically in all directions because the problem is angularly symmetric. We assume the plane is grounded symmetrically at infinity, allowing the current to flow away. For now, we do not consider another conductor making contact. Since the current spreads out to infinity, its surface density must decrease with distance. Additionally, conservation of current dictates that the total current crossing an imaginary circle of radius r centered at the origin must satisfy

$$2\pi r j_\mathrm{s}(r)=I\,.$$

However, we must work with vectors. In the previous equation, $j_\mathrm{s}(r)$ represents the magnitude of the surface current density at a distance r from the center, and this relationship holds only because the current density vector is perpendicular to the boundary of the imaginary circle. Knowing both the magnitude and direction of this vector, we can express its components as

$$j_{\mathrm{s},x}=\frac{I}{2\pi}\frac{x}{x^2+y^2}\,,\quad j_{\mathrm{s},y}=\frac{I}{2\pi}\frac{y}{x^2+y^2}\,.$$

The expressions above describe the current density generated by current I flowing into the plane at the origin. Now, by the principle of superposition, we add the effect of the current flowing out at $[a, 0]$, which has the same form but with an opposite sign (since it flows outward) and shifted in x

$$j_{s,x} = -\frac{I}{2\pi} \frac{x - a}{(x-a)^2 + y^2}, \quad j_{s,y} = -\frac{I}{2\pi} \frac{y}{(x-a)^2 + y^2}.$$

Using the notation where the incoming current has a superscript *in* and the outgoing one *out*, the total surface current density is

$$\mathbf{j}_s = \mathbf{j}_s^{in} + \mathbf{j}_s^{out} = \frac{I}{2\pi} \left(\frac{x}{x^2+y^2} - \frac{x-a}{(x-a)^2+y^2}, \frac{y}{x^2+y^2} - \frac{y}{(x-a)^2+y^2} \right).$$

Now that we have the current density, we can use the first equation to express the electric field. The voltage between the measurement electrodes is given by

$$\int_{[0,a]}^{[a,a]} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \,,$$

where $\mathrm{d}\mathbf{l}$ is the tangent vector along the path connecting these two points. We choose a straight-line path. Due to the dot product in the integral, we only need to consider the x-component of the electric field, as the y-component is perpendicular to this path. Since all points on the path have the same y-coordinate, we can substitute $y = a$ as we integrate over x from 0 to a

$$U = \frac{I}{2\pi\sigma_s} \int_0^a \left(\frac{x}{x^2+a^2} - \frac{x-a}{(x-a)^2+a^2} \right) \mathrm{d}x \,.$$

Using the substitution $u = x^2 + a^2, \mathrm{d}u = 2x\,\mathrm{d}x$, we compute the first term in the integral as

$$\int_0^a \frac{x}{x^2+a^2} \,\mathrm{d}x = \int_{a^2}^{2a^2} \frac{1}{2u} \,\mathrm{d}u = \frac{1}{2} \ln\left(\frac{2a^2}{a^2}\right) = \frac{1}{2} \ln 2 \,.$$

By symmetry, the second term in the integral gives the same result with an opposite sign. Thus, the voltage-current relationship gets simplified to

$$U = \frac{I}{2\pi\sigma_s} \left(\frac{1}{2} \ln 2 - \left(-\frac{1}{2} \ln 2 \right) \right) = \frac{\ln 2}{2\pi} \frac{I}{\sigma_s} \,,$$

from which we solve for the surface conductivity

$$\sigma_s = \frac{\ln 2}{2\pi} \frac{I}{U} \doteq 0.110\,3 \frac{I}{U} \,.$$

In our solution, we assumed a sign convention where both current and voltage are positive. Since the problem does not specify this convention, we could include absolute values for I and U to ensure a positive conductivity. However, this is not strictly necessary for the correct answer.

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Problem GG ... cylinder à la rocket

Consider a hollow cylinder with base radius $r = 3.00\,\mathrm{cm}$ *and height* $H = 30.0\,\mathrm{cm}$, *which is half filled with water. The cylinder without water has a mass* $M = 200\,\mathrm{g}$ *and stands on a base (its base is horizontal) on which it can move without friction. Just above the base of the cylinder there is a small hole with a diameter* $r_0 = 0.500\,\mathrm{mm}$, *from which water flows out perpendicularly to the surface of the cylinder. What will be the speed of the cylinder after all the water flows out? Assume that the acceleration of the cylinder is small enough so that the outflow velocity will be given by the Torricelli relation.* *Rado's morning coffee has escaped from him.*

Since the hole in the wall of the container is small compared to the water surface and the inertial force on the water can be neglected, the outflow velocity u is given by the Torricelli law

$$u = \sqrt{2gh}\,,$$

where h denotes the height of the liquid surface in the container. Now let us denote the mass of the cylinder M and the mass of water remaining in the cylinder at that moment m. Next, suppose that the cylinder is moving at a given moment with a velocity v, then from the law of conservation of momentum we have

$$(M + m)\vec{v} = (M + m - \Delta m)(\vec{v} + \mathrm{d}\vec{v}) + \Delta m(\vec{u} + \vec{v})\,.$$

By simplifying and neglecting the term $\mathrm{d}m\,\mathrm{d}\vec{v}$ we get

$$0 = (M + m)\,\mathrm{d}\vec{v} + \Delta m\vec{u}\,.$$

Since $\vec{u}$ and $\mathrm{d}\vec{v}$ have opposite directions, we can rewrite the equation in the form

$$(M + m)\,\mathrm{d}v = \Delta m u\,.$$

Now we should also note that Δm denotes the mass of the spilled water (i.e. the negative sign when added to the mass of the water in the container m), so $\Delta m = -\mathrm{dm}$.

$$\mathrm{d}v = -\frac{\mathrm{d}m}{M + m}u$$

Next, we need to express u in terms of m. So we express h as a function of m and plug it into the Toricelli relation

$$h = \frac{m}{\rho S}\,,$$

where S is the surface of the water and ρ is the density of the water. By substituting it, we get

$$u = \sqrt{2g\frac{m}{\rho S}}\,.$$

And thus

$$\mathrm{d}v = -\sqrt{\frac{2g}{\rho S}}\frac{\sqrt{m}}{M + m}\,\mathrm{d}m\,.$$

By integrating, we get

$$v_{\max} = -\sqrt{\frac{2g}{\rho S}}\int_{m_0}^{0}\frac{\sqrt{m}}{M + m}\,\mathrm{d}m\,.$$

The integral on the right hand side can be solved by substituting for $\sqrt{m}$

$$\int \frac{\sqrt{m}}{M+m}\,\mathrm{d}m = 2\int \frac{t^2}{M+t^2}\,\mathrm{d}t = 2\int \mathrm{d}t - 2\int \frac{1}{\left(\frac{1}{\sqrt{M}}t\right)^2+1}\,\mathrm{d}t = 2t - 2\sqrt{M}\arctan\left(\frac{1}{\sqrt{M}}t\right).$$

For t we substitute $\sqrt{m}$ and we obtain the solution

$$v_{\max} = -2\sqrt{\frac{2g}{\rho S}}\left[\sqrt{m} - \sqrt{M}\arctan\left(\sqrt{\frac{m}{M}}\right)\right]_{m_0}^{0},$$

$$v_{\max} = 2\sqrt{\frac{2g}{\rho S}}\left(\sqrt{m_0} - \sqrt{M}\arctan\left(\sqrt{\frac{m_0}{M}}\right)\right),$$

where m_0 is the original mass of water in the cylinder. Expressing S and m_0 using the given quantities gives us

$$S = \pi r^2,$$

$$m_0 = \frac{H}{2}\pi r^2 \rho,$$

$$v_{\max} = 2\sqrt{\frac{2g}{\rho\pi r^2}}\left(\sqrt{\frac{H}{2}\rho\pi r^2} - \sqrt{M}\arctan\left(\sqrt{\frac{\frac{H}{2}\rho\pi r^2}{M}}\right)\right),$$

$$v_{\max} \doteq 1.15\,\mathrm{m{\cdot}s^{-1}}.$$

The maximum velocity the cylinder will reach is therefore $1.15\,\mathrm{m{\cdot}s^{-1}}$.

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Problem GH ... a little restive dipole

A little magnetic dipole with the magnetic moment $m = 1.0\cdot 10^3\,\mathrm{A{\cdot}m^2}$ harmonically oscillates on the axis of a conductive circular loop with the radius $R = 1.0\,\mathrm{m}$, at a frequency $f = 1.0\,\mathrm{MHz}$ and with an amplitude of the deflection equal to $h_0 = 1.0\,\mathrm{mm}$. The direction of the magnetic moment is parallel to the axis of the loop and the equilibrium position of the little dipole is located in the geometric center of the loop (meaning the little dipole oscillates between the maximum distance h_0 below and h_0 above the loop). Determine the amplitude of the induced voltage in the loop, assuming the little dipole is sufficiently small, and therefore the vector potential it creates can be expressed as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3},$$

where $\mathbf{r}$ is the vector from the little dipole to any point in space, $r = |\mathbf{r}|$, and μ_0 is the vacuum permeability. Recall that the magnetic induction is given by $\mathbf{B} = \nabla\times\mathbf{A}$.

Kuba likes little dipoles.

Let us count a magnetic induction flux Φ generated by the little dipole inside the loop. From the formula for magnetic induction in the assignment and from Stokes' integral theorem we recieve

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot \mathrm{d}\mathbf{l} \,.$$

Let the equilibrium position of the little dipole be located in the origin of a Cartesian coordinate system and the magnetic moment heads in a positive direction of the axis z. Then

$$\mathbf{r} = \mathbf{R} + \mathbf{h} \,,$$

where $\mathbf{R}$ is the vector from the middle of the loop to the point on the loop and $\mathbf{h}$ is the vector from the little dipole to the middle of the loop. Count

$$\Phi = \oint_{\partial S} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \frac{\mu_0}{4\pi r^3} \oint_{\partial S} \mathbf{m} \times (\mathbf{R} + \mathbf{h}) \cdot \mathrm{d}\mathbf{l} \,.$$

In front of the integral we put the total distance r of the little dipole from the point on the loop, because it is the same for every point of the loop. Since $\mathbf{m} \perp \mathbf{R}$ and $\mathbf{m} \parallel \mathbf{h}$, we receive

$$\Phi = \frac{\mu_0}{4\pi r^3} \oint_{\partial S} mR \,\mathrm{d}l = \frac{\mu_0}{4\pi r^3} 2\pi R \, mR = \frac{\mu_0 m R^2}{2r^3} = \frac{\mu_0 m R^2}{2\left(R^2 + h^2\right)^{3/2}} \,.$$

Now let us consider time dependenece

$$h = h_0 \cos(\omega t) \,.$$

For the induced voltage U we receive

$$\begin{aligned} U &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \,, \\ &= -\frac{\mu_0 m R^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[R^2 + h^2\right]^{-3/2} \,, \\ &= \frac{3\mu_0 m R^2}{4} \left[R^2 + h^2\right]^{-5/2} 2h \frac{\mathrm{d}h}{\mathrm{d}t} \,, \\ &= -\frac{3\mu_0 m R^2}{2} \left[R^2 + h_0^2 \, \cos^2(\omega t)\right]^{-5/2} h_0^2 \, \omega \cos \omega t \sin \omega t \,. \end{aligned} \tag{6}$$

Now we use the formula

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \,, \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

and the equation (6) we will rewrite in the form

$$U = -\frac{3\mu_0 m R^2}{4} \left[R^2 + \frac{h_0^2}{2} + \frac{h_0^2 \cos(2\omega t)}{2}\right]^{-5/2} h_0^2 \omega \sin(2\omega t) \,. \tag{7}$$

From the assignment it is evident that $h_0 \ll R$. So let us develop the result to a Taylor expansion for $\varepsilon = h_0/R$

$$
\begin{aligned}
U &= -\frac{3\mu_0 m\omega R^2 h_0^2}{4}\left[R^2+\frac{h_0^2}{2}+\frac{h_0^2\cos(2\omega t)}{2}\right]^{-5/2}\sin(2\omega t)\\
&= -\frac{3\mu_0 m\omega \varepsilon^2}{4R}\left[1+\varepsilon^2\frac{1+\cos(2\omega t)}{2}\right]^{-5/2}\sin(2\omega t)\\
&= -\frac{3\mu_0 m\omega \varepsilon^2}{4R}\left[1-\frac{5}{2}\frac{1+\cos(2\omega t)}{2}\varepsilon^2+\mathcal{O}(\varepsilon^4)\right]\sin(2\omega t)\\
&= -\frac{3\mu_0 m\omega \varepsilon^2}{4R}\sin(2\omega t)+\mathcal{O}(\varepsilon^4)\,.
\end{aligned}
$$

Since h_0 is three orders less than R, additional members will have no effect on the solution given (in accordance with the assignment) with precision to two valid numbers.

For the amplitude of voltage U_0 we can write

$$
U_0 \approx \frac{3\mu_0 m\omega h_0^2}{4R^3} = \frac{3\pi}{2}\frac{\mu_0 m f h_0^2}{R^3} \doteq 5.9\,\mathrm{mV}\,.
$$

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