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Solutions

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Solutions of problems

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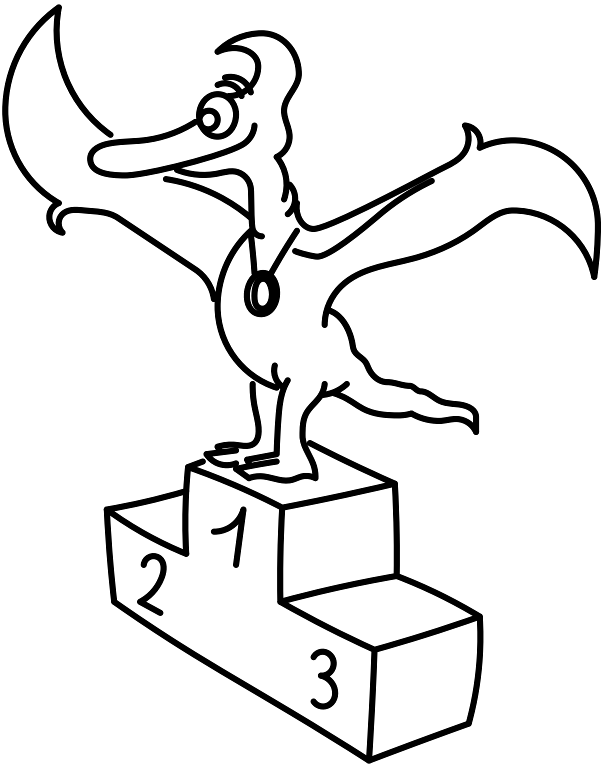

Problem AA ... at the concert

Danka was at the concert at the airport. During one song, they had the drumming and the flashing of the spotlights over the stage synchronized. It was a periodic drumming and flashing with a period of $T = 1.5\,\mathrm{s}$, and the two events always happened simultaneously. However, Danka saw that the spotlights were flashing with a half-period offset from the sound of the drums. What is the smallest possible distance Danka could be from the stage to observe this phenomenon? *Danka and other organizers were at the Rammstein concert.*

When Danka stands at the distance x from the stage, the light from the stage reaches her in $t_1 = x/c$, where c is the speed of light. Similarly, the drumming propagates through the air to Danka at the speed of sound in the air v, so the sound wave reaches her in $t_2 = x/v$. Since Danka sees that the light and sound waves reach her with a $T/2$ offset, the following must hold

$$t_2 - t_1 = \frac{T}{2}\,.$$

We insert the above-mentioned formulas for the times t_1 and t_2 and then express the distance we are looking for

$$\frac{x}{v} - \frac{x}{c} = \frac{T}{2}\,,$$
$$x\left(\frac{c-v}{cv}\right) = \frac{T}{2}\,,$$
$$x = \frac{T}{2}\left(\frac{cv}{c-v}\right)\,.$$

Now, we can notice that the second fraction in the last equation can be modified to the form $v/(1 - v/c)$, and since the v/c ratio is several orders of magnitude smaller than 1, the whole fraction is quite exactly equal to v. Then

$$x = \frac{vT}{2} \doteq 257\,\mathrm{m}\,.$$

Hence, Danka had to stand at a distance of $257\,\mathrm{m}$ from the stage. In general, Danka can stand at the places that satisfy the condition $t_2 - t_1 = (2n - 1) \cdot T/2$, where n is a natural number. However, since we are interested in the smallest distance, we consider $n = 1$ in the whole solution.

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Problem AB ... slow cyclist

Verča is driving her car on the road at speed $v_1 = 82\,\mathrm{km{\cdot}h^{-1}}$. A cyclist is pedaling along the side of the road at speed $v_2 = 16\,\mathrm{km{\cdot}h^{-1}}$ and Verča wants to pass him, so she has to drive into the middle of the road. As she does not want to endanger him, she leaves the lane $d = 20\,\mathrm{m}$ before the cyclist and always returns back at a distance $d = 20\,\mathrm{m}$ after him (in the direction of travel). How long would a stationary obstacle on the side of the road have to be for Verča to spend the same amount of time going around it as she did when passing the cyclist? She only goes around stationary things with a margin of $l = 10\,\mathrm{m}$. Ignore the time required to cross *between the center of the road and the lane. Think of the car and the cyclist as points (do not consider their length).* *Verča doesn't like going around obstacles. So she doesn't drive.*

Let us denote the length of the obstacle we are looking for as S and the distance the cyclist will travel while the car is in the passing lane as s. The key to solving the problem is to express the time t that the car spends here. From the above, we get the equation

$$v_2 \cdot t = s\,.$$

The second equation in the system describes the distance the car travels in time t, i.e.

$$v_1 \cdot t = 2d + s\,,$$

because the car goes around the cyclist with a margin d on both sides.

From this system of equations, we can easily express the time t and distance s as

$$s = \frac{2d}{\frac{v_1}{v_2} - 1}\,, \quad t = \frac{2d}{v_1 - v_2}\,.$$

The distance S we are looking for is this cyclist's path, to which we add the difference in the overtaking margin, so we get

$$S = s + 2(d - l) = \frac{2d}{\frac{v_1}{v_2} - 1} + 2(d - l)\,.$$

After plugging in the numerical values, we find that the obstacle would have to measure approximately $S \doteq 30\,\mathrm{m}$.

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Problem AC ... not enough time

Daniel needs to extend the time he has left to write his diploma thesis. Ideally, in a way that three weeks becomes six. The simplest and best solution seems to be to move the Earth by a little. By how many astronomical units does Daniel need to increase the Earth's average distance from the Sun to double the Earth's orbital period? Write the answer with 3 significant digits. *Daniel needs more time to write his diploma thesis.*

We will use the simplified Kepler's Third Law, where $a^3 = P^2$ holds for the orbital period P in years and the average distance from the Sun[1] a in astronomical units. For the new orbital period $P = 2\,\mathrm{years}$, we will get the equation $a^3 = 4$, from which we can take the cube root and get $a = 1.587\,\mathrm{au}$. If the average Earth-Sun distance is $1\,\mathrm{au}$, Daniel will have to move the Earth by approximately 0.587 astronomical units, which is beyond the orbit of Mars.

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[1] more precisely, the length of the orbital semi-major axis

Problem AD ... springs and weights

We have three identical springs of negligible mass with stiffness k and three weights of equal mass m. We attach one spring to the ceiling and hang one of the weights on its other end. To this weight, we add another spring with a weight on its tail and finally a third spring and a third weight. By how much do springs elongate with respect to their rest length?

Karel reminisced about springs.

We solve the elongation of each spring separately and then add them up. If we index springs from the bottom one, the elongation of the first spring is

$$F_1 = mg = ky_1 \quad \Rightarrow \quad y_1 = \frac{mg}{k}\,.$$

Analogically for the second and third one

$$y_2 = \frac{2mg}{k}\,, \quad y_3 = \frac{3mg}{k}\,.$$

Then our result is

$$\Delta y = y_1 + y_2 + y_3 = \frac{6mg}{k}\,.$$

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Problem AE ... gramophone

The camera records a vinyl record, which is symmetrically crossed by six lines running centrally from one end to the other. The display shows the record starting to spin gradually. At a certain point, it seemed that it had stopped moving. At that moment, the record's perimeter rotates with a speed $v = 3.14\,\mathrm{m\cdot s^{-1}}$, while the record's radius is $r = 10\,\mathrm{cm}$. Determine the frame rate of the camera.

The promising FYKOS-bird forgot to blink.

The angle between two lines on the board is

$$\alpha = \frac{\pi}{6}\,.$$

The record looks as if it stopped on display when the record rotates between two frames by any multiple of this angle. However, the plate is gradually rotating, so we are looking for the smallest rotation, and therefore the frame rate is

$$f = \frac{\omega}{\alpha} = \frac{6v}{\pi r} \doteq 60\,\mathrm{s^{-1}}\,.$$

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Problem AF ... ball on the boat

Lego and Dodo are sailing down the river on a boat and throwing a ball around on the deck. They both stand exactly parallel to the boat's course and the water in the river. Robo, standing on the shore, watches them from afar. When Lego throws the ball to Dodo, Robo sees that the ball has a horizontal velocity $v_1 = 42\,\mathrm{km\cdot h^{-1}}$*, when Dodo throws it to Lego, Robo observes a velocity* $v_2 = 24\,\mathrm{km\cdot h^{-1}}$ *in the opposite direction. Lego and Dodo confirm to Robo that they both throw at the same horizontal speed. At what velocity is the ship sailing relative to Robo, and in which direction (i.e., from Dodo to Lego or from Lego to Dodo)?*

Karel wanted to trump Nanynka's cabbage problem.

Let's denote the boat's velocity v_L and the speed at which Lego and Dodo throw (i.e., the ball's speed relative to the boat) as v_H. Of the velocities that Robo observes, v_1 is the larger one. He observes this when the ball is thrown in the direction of the boat's motion, so the velocities will add $v_1 = v_\mathrm{L} + v_\mathrm{H}$. From the fact that Robo observes this velocity when Lego throws to Dodo, we can also see that the boat is sailing away from Lego toward Dodo.

The velocity v_2 is observed when the ball is thrown in the opposite direction to the sail. The magnitude of this velocity will be the difference in magnitudes of v_L and v_H, so $v_2 = |v_\mathrm{L} - v_\mathrm{H}|$. We still need to figure out which of the two velocities is larger to eliminate the absolute value. From the problem statement, the velocity v_2 is observed by Robo in the opposite direction to v_1. This is only possible if Lego and Dodo are tossing each other at a velocity greater than the boat's velocity, so $v_2 = v_\mathrm{H} - v_\mathrm{L}$ holds.

In summary, we have a system of equations

$$v_1 = v_\mathrm{H} + v_\mathrm{L}\,,$$
$$v_2 = v_\mathrm{H} - v_\mathrm{L}\,,$$

where the unknowns are v_H and v_L. However, we are only interested in the speed of the boat, so it is sufficient to subtract the second equation from the first to get

$$v_1 - v_2 = 2v_L\,,$$
$$v_L = \frac{v_1 - v_2}{2} = 9\,\mathrm{km\cdot h^{-1}}\,,$$

so the answer to the question is that the ship is sailing in the direction from Lego to Dodo at $9\,\mathrm{km\cdot h^{-1}}$.

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Problem AG ... weighing a dog

Jarda went with his dog to the vet, where he put him on a scale and weighed him. Once he read his mass, he pulled the leash, but the dog did not move at all. The scale showed a 10 percent lower reading than before. What is the minimal coefficient of friction between the scale and the dog's paws? Jarda pulled the leash at an angle of 40° *with respect to the ground.*

A final tribute to Dort the dog.

Let us denote the magnitude of force by F, the mass of the dog by m, and the angle by $\alpha = 40^\circ$. When weighing with a taut leash, a normal force acts on the scale

$$F_\mathrm{N} = mg - F\sin\alpha\,,$$

so according to the problem statement $F\sin\alpha = 0.1mg$.

In the horizontal direction, the force $F\cos\alpha$ acts against the friction force, which is directly proportional to the coefficient of friction f as $F_\mathrm{t} = fF_\mathrm{N}$. The dog did not slip on the scale, so it must hold $F_\mathrm{t} > F\cos\alpha$, from which

$$f > \frac{F\cos\alpha}{mg - F\sin\alpha}\,.$$

Now we just substitute for F from the second equation and get

$$f > \frac{\cos\alpha}{9\sin\alpha} = 0.13\,.$$

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Problem AH ... filling a bucket

A garden hose has a length of $15\,\mathrm{m}$ *and an inner diameter of* $1.5\,\mathrm{cm}$*. The* $11\,\mathrm{m}$ *long first part of the hose lies in direct sunlight; the rest leads through the shade to the tap. The sun beams warmed the water in the first part to* $35\,\mathrm{^\circ C}$*, while a* $15\,\mathrm{^\circ C}$ *water flows from the tap. Assume that between the warmed-up part of the hose and the tap, the water temperature changes linearly. Now, we begin to fill up the bucket from the hose. What temperature will the water have at the end if we fill the bucket with* $5.5\,\ell$ *of water?*

Jarda reminisces his garden and warm summer days.

A volume of water $V_1 = l_1 S \doteq 1.9\,\ell$ lies in the hose in the direct sunlight, where $l_1 = 11\,\mathrm{m}$ a $S = \pi d^2/4$ is the cross-sectional area of the hose, with $d = 1.5\,\mathrm{cm}$. This water has a temperature of $t_1 = 35\,\mathrm{^\circ C}$. Let us calculate the heat stored in it. Since heat is an additive quantity, let us set the zero heat level of water at $0\,\mathrm{^\circ C}$. Thus, after the subsequent calculation, we get the temperature in degrees Celsius.

$$Q_1 = l_1 S\rho c t_1\,,$$

where c is the specific heat capacity of water, and ρ is its density.

In the second part of the hose, the temperature changes linearly between $35\,\mathrm{^\circ C}$ and $t_2 = 15\,\mathrm{^\circ C}$, which corresponds to an average temperature $t_\mathrm{p} = (t_1 + t_2)/2$. Thus, the heat of this part is

$$Q_2 = l_2 S\rho c\frac{t_1 + t_2}{2}\,,$$

where $l_2 = l - l_1 = 11\,\mathrm{m}$.

Since $Sl \doteq 2.6\,\ell$ is still less than $V = 5.5\,\ell$, we also have to fill the bucket with water that has not gone through the tap yet. We need $V - lS$ of this water. Its heat is

$$Q_3 = (V - lS)\,\rho c t_2\,.$$

Since the fluid in the bucket is homogeneous, we obtain its temperature in degrees Celsius as the ratio of total heat to total mass and specific heat capacity as

$$t = \frac{1}{\rho c}\frac{Q_1 + Q_2 + Q_3}{V_1 + V_2 + V_3} = \frac{l_1 S t_1 + (l - l_1)\,S\,\frac{t_1 + t_2}{2} + (V - lS)\,t_2}{V} = 23.4\,\mathrm{^\circ C}\,.$$

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Problem BA ... CR7

Perhaps the best footballer in the world, Cristiano Ronaldo, is 187 cm *tall. However, in header duel, he can jump so high that the top of his head is at a height of* 268 cm*. The timing of the jump is very important to score a header. What percentage of the total airtime is a part of his head above the* 250 cm *level, where he can hit a flying ball?* *Jarda still thinks Messi is better.*

Let us denote Ronaldo's height by h_R, the height he can jump to h_v, and the height where he needs to hit the ball by $h_\mathrm{b} = 250\,\mathrm{cm}$. Ronaldo lifts by unbelievable $h_\mathrm{v} - h_\mathrm{R} = 81\,\mathrm{cm}$ during his jump. According to laws of motion in a homogeneous gravitational field, which holds on a pitch, he spends the time

$$t_\mathrm{v} = 2\sqrt{\frac{2\left(h_\mathrm{v} - h_\mathrm{R}\right)}{g}}$$

in the air, where g is the gravity of Earth.

Similarly, we can express a condition when he is higher than h_b. That gives us time

$$t_\mathrm{p} = 2\sqrt{\frac{2\left(h_\mathrm{v} - h_\mathrm{b}\right)}{g}}\,.$$

We are interested in the ratio

$$\frac{t_\mathrm{p}}{t_\mathrm{v}} = \sqrt{\frac{h_\mathrm{v} - h_\mathrm{b}}{h_\mathrm{v} - h_\mathrm{R}}} = 0.47\,,$$

which is 47 %. Thus, for almost half the time he is in the air, Ronaldo can hit a ball flying at 250 cm.

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Problem BB ... chopping a parsley

Jarda chopped the parsley, which had the shape of a perfect cone with an apex angle of $2\alpha = 10\,^\circ$*. He got bored of cutting all the pieces with the same width, so he started chopping them into pieces with constant volume* $V = 0.9\,\mathrm{cm}^3$*. What is the width of the seventh piece if he starts slicing from the tip?* *A large piece of vegetable was floating in Jarda's soup.*

Let us denote the distance from the tip to the nearer plane of the n-th circle as a h_{n-1}, and the distance of the tip from the second slice as a h_n. Therefore h_0 is equal to zero.

The volume of the first piece is V, which shape is a cone. Its volume is

$$V = \frac{1}{3}\pi r^2 h_1 = \frac{1}{3}\pi h_1^3 \tan^2 \alpha\,,$$

where we expressed the radius of the base using the apex angle and height.

The volume of the first n pieces is nV, so the total height is

$$h_n = \sqrt[3]{\frac{3nV}{\pi \tan^2 \alpha}}\,.$$

Subtracting the h_6 from the h_7 will give us the requested result (remember that the $\alpha = 5\,^\circ$)

$$t_n = \sqrt[3]{\frac{3V}{\pi \tan^2 \alpha}} \left(\sqrt[3]{n} - \sqrt[3]{n-1}\right) = 0.46\,\mathrm{cm}\,.$$

Obviously, the width of the discs is decreasing.

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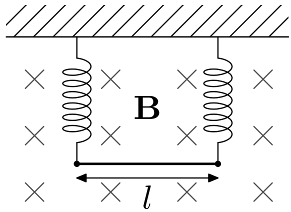
Problem BC ... a rod on springs



A metal rod with a length $l = 34\,\mathrm{cm}$ and a mass $m = 85\,\mathrm{g}$ is suspended by its ends from the ceiling using two conductive springs. We create a homogeneous magnetic field of size $B = 0.440\,\mathrm{T}$ oriented as is shown in the figure. How much current does it need to be running through the rod in the right direction to have the springs not stretched at all?
Danka remembered the time she was studying electromagnetism.

In order for the springs not to be stretched at all, the gravitational force acting on the rod $F_\mathrm{G} = mg$ in the downward direction must be offset by another force acting upward. When we insert a conductor through which an electric current is flowing into a magnetic field, a magnetic force will start to act on it. Its magnitude is described by Ampere's law

$$F_\mathrm{m} = IlB \sin \alpha\,,$$

where α is the angle between the direction of the magnetic field and the direction of the current. In our case, the rod is perpendicular to the magnetic field, so $\alpha = 90\,^\circ$, and therefore $\sin \alpha = 1$. When the magnetic and gravitational forces are equal, the following equation holds

$$mg = IlB \sin \alpha\,.$$

From here, we express the magnitude of the current we are looking for and plug in the numerical values

$$I = \frac{mg}{lB \sin \alpha} = \frac{mg}{lB} \doteq 5.6\,\mathrm{A}\,.$$

Therefore a current of 5.6 A has to be flowing through the rod.

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Problem BD ... a swing

The swing chain deviates from the vertical line at the highest point by $\alpha = 70\,^\circ$. When the seat of the swing passes through the lowest point, the chain suddenly breaks. The chain is $l = 2\,\mathrm{m}$ long, and the seat is at rest at the height $h_0 = 1\,\mathrm{m}$ above the ground. How far does the seat fall from the vertical line of the swing? *A promising FYKOS-bird is thinking about the future.*

First, we determine the maximum height that the swing can reach

$$h = h_0 + l\left(1 - \cos\alpha\right) .$$

Since we assume that mechanical energy is being conserved ($E_\mathrm{k} = E_\mathrm{p}$), we can determine the horizontal component of the velocity as

$$\frac{1}{2}mv_x^2 = mg\left(h - h_0\right) ,$$

from which we express the horizontal velocity

$$v_x = \sqrt{2gl\left(1 - \cos\alpha\right)} .$$

We write the height of the vertical throw in a homogeneous gravitational field as $h_0 = gt^2/2$. From this, we find the time

$$t = \sqrt{\frac{2h_0}{g}} ,$$

which, when multiplied by the formula for v_x, gives us the horizontal distance we are looking for

$$s_x = v_x t = \sqrt{4h_0 l\left(1 - \cos\alpha\right)} = 2.3\,\mathrm{m} .$$

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Problem BE ... dense energy

Nowadays, we are looking for ways to get energy. One way is to generate electricity from renewable sources such as the wind or the sun. However, these do not produce electricity under all conditions, so the energy needs to be stored. Let us compare two options – pumped hydroelectric energy storage and compressed hydrogen. The Dlouhé stráně power plant has a working volume of water in the upper reservoir of $2\,580\,000\,\mathrm{m}^3$*, a water gradient of* $510\,\mathrm{m}$ *and an efficiency of* $90\,\%$*. Hydrogen gas* H_2 *can be stored compressed in* $50\,\ell$ *cylinders at a pressure of* $70\,\mathrm{MPa}$*. The heating value of hydrogen is* $120\,\mathrm{MJ{\cdot}kg^{-1}}$ *and the conversion efficiency to electricity is* $50\,\%$*. Consider hydrogen an ideal gas with a temperature of* $25\,\mathrm{^\circ C}$*. How many of these filled hydrogen cylinders are equivalent to the available electricity of the Dlouhé stráně power plant?* *Jarda wanted to stock up electricity before it gets more expensive.*

The available energy at a pumped storage power plant is the potential energy of a homogeneous gravity field

$$E_\mathrm{e} = \eta_\mathrm{e} V_\mathrm{e} \rho_\mathrm{v} g h = 11\,600\,\mathrm{GJ} ,$$

where $\eta_\mathrm{e} = 0.9$, V_e is the volume of the upper reservoir, ρ_v is the density of the water, and $h = 510\,\mathrm{m}$ is the gradient of the water.

We calculate the energy in one bottle of hydrogen from its mass and gravimetric energy (energy stored per unit mass). Considering hydrogen is an ideal gas, its density at pressure $p = 70\,\mathrm{MPa}$ and temperature $T = 298\,\mathrm{K}$ is

$$\rho_\mathrm{h} = \frac{M_\mathrm{m} p}{TR} = 56.5\,\mathrm{kg{\cdot}m^{-3}} ,$$

where $M_\mathrm{m} \doteq 2\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of the hydrogen molecule.

The available energy from a bottle of hydrogen is thus

$$E_\mathrm{h} = \eta_\mathrm{h} l V_\mathrm{l} \rho_\mathrm{h} = 170\,\mathrm{MJ}\,,$$

where $\eta_\mathrm{h} = 0.5$, l is the gravimetric energy and V_l is the volume of the bottle.

By dividing the two energies, we get the number of bottles $N = 68\,000$. However, the total volume of hydrogen in all the bottles is several orders of magnitude smaller than the volume of the hydropower plant reservoir.

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Problem BF ... a pulley that shifts

We have a massless movable pulley. One end of a massless rope that goes through the pulley is attached directly to the ceiling; the other end is connected to a spring with stiffness $k = 80\,\mathrm{N{\cdot}m^{-1}}$, which is attached to the same ceiling. We will hang a weight with mass $m = 1.0\,\mathrm{kg}$ onto the pulley. How far will the pulley shift downwards?

This crossed Lego's mind when he was writing down a different problem...

The whole system will be at equilibrium when both sides of the rope pull the pulley with force $mg/2$. The spring will be strained by $mg/(2k)$ with respect to its initial length. The extensional strain is distributed equally between the two halves of the rope coming out of the pulley; thus, the pulley will only move by $mg/(4k) = 3.1\,\mathrm{cm}$.

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Problem BG ... does is float?

We have a cube, and the only thing you know about it is that its sides are shorter than $10\,\mathrm{cm}$. Furthermore, we have a container with a square base of side $a = 10\,\mathrm{cm}$, which we fill with water of volume $V = 1\,\ell$. When we drop our cube into the container, the level of water rises by $\Delta h_1 = 3.5\,\mathrm{cm}$. We then remove the cube, pour the water out, and replace it with methanol of volume V and density $\rho_\mathrm{M} = 792\,\mathrm{kg{\cdot}m^{-3}}$. When we drop the cube into the container with methanol, the level rises by $\Delta h_2 = 4.2\,\mathrm{cm}$. What is the density of our cube? Yes, you do have all the data needed.

Lego wanted to assign an interesting Archimedes problem.

The key observation in this problem is that $\Delta h_1 \rho_\mathrm{V} \neq \Delta h_2 \rho_\mathrm{M}$, i.e. in at least one liquid the cube lies at the bottom of the container. If that wasn't the case, and the cube was indeed floating in both liquids, the buoyant force would have to be equal in both cases, thus $a^2 \Delta h_1 \rho_\mathrm{V} g = a^2 \Delta h_2 \rho_\mathrm{M} g$. Divide the previous expression by $a^2 g$ and we get a condition on the product of the rise of the level and density, which is not satisfied, therefore it cannot be true that the cube floats in both liquids.

Moreover, it cannot lie on the bottom of the container in both liquids, since $\Delta h_1 \neq \Delta h_2$. Thus, it obviously floats in one liquid and lies on the bottom in the other. Logically, the cube will float in the liquid with higher density and lie on the bottom in the one with the lower density.

From the rise of the methanol level, we can easily calculate the volume of the cube as $V_\mathrm{k} = a^2\Delta h_2$. We can also calculate the mass of the cube from the rise of the water level since its weight must be the same as the weight of a liquid whose volume is equal to the volume of the submerged part of the cube, i.e. $m_\mathrm{k} = a^2\Delta h_1\rho_\mathrm{V}$.

The density of the cube will therefore be

$$\rho_\mathrm{k} = \frac{m_\mathrm{k}}{V_\mathrm{k}} = \frac{a^2\Delta h_1\rho_\mathrm{V}}{a^2\Delta h_2} = \frac{\Delta h_1}{\Delta h_2}\rho_\mathrm{V} = 832\,\mathrm{kg{\cdot}m^{-3}}\,.$$

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Problem BH ... restless coin

When riding up the inclined moving walkway of inclination $\alpha = 10^\circ$ and length $l = 30\,\mathrm{m}$, a coin drops out of Verča's pocket when she is exactly in the middle of it. It falls into one of the grooves on the walkway and starts rolling down without slipping. How much time does Verča have to catch the coin before it falls under the bottom edge of the walkway? The velocity of the moving walkway is $v = 0.9\,\mathrm{m{\cdot}s^{-1}}$.

Verča is constantly losing the content of her pockets and fears the escalators.

First, we express the distance of the coin from the lower end of the walkway in terms of time. We determine it from the total energy balance for a decrease in height h. Potential energy converts to translational energy

$$E_\mathrm{k,t} = \frac{1}{2}mv^2$$

and rotational energy

$$E_\mathrm{k,r} = \frac{1}{2}J\omega^2\,.$$

Letter J denotes the moment of inertia. In our particular case, for a coin shaped like a cylinder, $J = mR^2/2$, where m is the mass of the coin and R is its radius. Angular velocity ω is bound to the velocity of the coin due to the no-slip condition as $v = R\omega$. Thus, adding the two contributions to the kinetic energy, we get the following from the law of conservation of energy

$$mg\Delta h = E_\mathrm{k,t} + E_\mathrm{k,r} = \frac{1}{2}mv^2 + \frac{1}{2}\frac{1}{2}m\left(R\omega\right)^2 = \frac{1}{2}\frac{3}{2}mv^2\,.$$

That is the same formula as for uniformly accelerated motion, except that the mass of the coin on the right is multiplied by a factor of 3/2. Thus, the coin will have a constant acceleration of magnitude

$$a = \frac{2}{3}g\sin\alpha\,.$$

The coin's motion down the walkway is uniformly accelerated, but its initial velocity v relative to the ground is in the opposite direction. So we get the equation

$$x = \frac{l}{2} + vt - \frac{1}{2}\frac{2}{3}gt^2\sin\alpha\,.$$

Putting $x = 0$ gives us the time in which the coin will reach the bottom edge. So, we need to solve the quadratic equation for t to get the roots

$$t_{1,2} = \frac{-v \pm \sqrt{v^2 + \frac{2}{3} l g \sin\alpha}}{-\frac{2}{3} g \sin\alpha} \,.$$

Putting in the numerical values, we get $t_1 \doteq -4.4\,\mathrm{s}$ and $t_2 \doteq 6.0\,\mathrm{s}$. We will not consider negative time, so Verča has roughly 6.0 seconds to get the coin.

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Problem CA ... oscillating pulley

Let us have a massless movable pulley. One end of a massless rope that goes through the pulley is attached directly to the ceiling, while the other end is attached to a spring of stiffness $k = 80\,\mathrm{N{\cdot}m^{-1}}$*, which is attached to the same ceiling. We hang a weight of mass* $m = 1.0\,\mathrm{kg}$ *on the pulley. What is the period of small oscillations of the weight?*

Lego loves pulleys and oscillators, so why not to combine them?!

When we deviate the weight from the equilibrium position lowering it by $\mathrm{d}x$, the pulley also moves by $\mathrm{d}x$. Thus, the total length of the rope and the spring must increase by $2\,\mathrm{d}x$. As the rope does not elongate, it must be the spring that is extended by this length. Thus, the spring will pull with a force $2k\,\mathrm{d}x$ greater than it was pulling while in the equilibrium position. Since the rope and the pulley are both massless, the force exerted by the spring is equal to the tension along the entire length of the rope. As the pulley is pulled upward by the rope on both sides (i.e. twice), the pulling force is $4k\,\mathrm{d}x$ greater than in the equilibrium position. The pulley is weightless, so this is also the contribution to the force, which is pulling our weight upward. When we divide the force contribution by the displacement that caused it, we get the effective stiffness that the weight experiences as $k_{\mathrm{ef}} = 4k$. (We reached the same conclusion in the problem "a pulley that shifts", where we found that, when a weight is hung on the pulley, the pulley moves down by $mg/(4k) = mg/k_{\mathrm{ef}}$.)

All that remains is to substitute into the formula for the period of a linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k_{\mathrm{ef}}}} = \pi\sqrt{\frac{m}{k}} = 0.35\,\mathrm{s}\,.$$

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Problem CB ... a crystal of beryl

Beryl crystallizes in a hexagonal crystal system, i.e., with the unit cell in the shape of a perpendicular prism, whose base is a rhombus. If we correctly join 3 unit cells, the resulting perpendicular prism has the base of a regular hexagon, with the side a. The height of the prism, c, is approximately equal to a. In the unit cell, there are atoms corresponding to two molecules of beryl, whose chemical formula is $\mathrm{Be_3Al_2(SiO_3)_6}$*. The relative atomic mass of beryllium is* $A_{\mathrm{Be}} = 9.01$*, aluminium* $A_{\mathrm{Al}} = 27.0$*, silicon* $A_{\mathrm{Si}} = 28.1$ *and oxygen* $A_{\mathrm{O}} = 16.0$*. What is* *the length of the hexagonal side a, if the density of beryl is $\rho = 2\,760\,\mathrm{kg{\cdot}m^{-3}}$?*

Karel made a simple high school problem more complicated.

Let us first calculate the mass of the beryllium molecule. We are given the relative atomic masses of all the atoms in the molecule. These give us the masses of the atoms as multiples of the atomic mass unit, which is defined as one-twelfth the mass of a carbon atom ^{12}C, which is $m_\mathrm{u} = 1.66 \cdot 10^{-27}$kg.

One molecule of beryl contains 3 atoms of Be, 2 Al, 6 Si, and $6 \cdot 3 = 18$ O, then the total mass of the molecule is

$$m_\mathrm{molecule} = (3A_\mathrm{Be} + 2A_\mathrm{Al} + 6A_\mathrm{Si} + 18A_\mathrm{O})m_u = 537,6m_u = 8,92 \cdot 10^{-25}\,\mathrm{kg}\,.$$

Next, we need to calculate the volume of one unit. The volume of a prism is the height times the area of the base; the height is simply a in our case. The base is a regular hexagon, so we calculate its area as 6 times the area of an equilateral triangle with side a

$$S_p = 6S_t = 3a\ v_a = 3a\sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \frac{3\sqrt{3}}{2}a^2\,,$$

and to get the volume, we need to multiply the calculated area by a.

All that is left to do is plug the obtained result in the formula for density, not forgetting that there are $2 \cdot 3$ molecules per one hexagonal base, and express a

$$\rho = \frac{m}{V} = \frac{6m_\mathrm{molecule}}{\frac{3\sqrt{3}}{2}a^3}\,,$$

$$a = \left(\frac{4m_\mathrm{molecule}}{\sqrt{3}\rho}\right)^{\frac{1}{3}} \approx 0.91\,\mathrm{nm}\,.$$

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Problem CC ... lasagna

Consider a layering lasagna. The first layer with everything in its place has a mass $m_0 = 100\,\mathrm{g}$ and has a temperature $T_0 = 300\,\mathrm{K}$. Each additional layer weighs $p = 1/3$ times more than the previous layer; however, it is fresher. That means its absolute temperature is $q = 2$ times higher than the previous layer. Since Pepa was very hungry, he stacked an infinite number of these layers on each other. At what temperature will the lasagna in its final form stabilize if the system reaches thermal equilibrium? Do not consider heat loss. Bon appétit.

Pepa was hungry.

First, let us calculate the weight of the whole lasagne. The weight of the ith layer will be $m_i = m_0p^i$. Therefore the mass of the whole lasagne will be

$$M = \sum_{i=0}^{\infty} m_0p^i\,.$$

That is a geometric series; those of you who know the formula for its sum, go ahead and use it; however, for those that do not know the formula, we will show you how to get it. The trick is to multiply both sides of the equation by p, and we will also substitute $j = i+1$ on the right-hand side

$$pM = \sum_{i=0}^{\infty} m_0 p^{i+1} = \sum_{j=1}^{\infty} m_0 p^j \,.$$

Notice that on the right-hand side, we have the same sum, which was equal to M, with the difference that it starts from the first term, not the zeroth term. Thus at the right-hand side, we have $M - m_0$, from which we get the linear equation

$$pM = M - m_0$$
$$M = \frac{m_0}{1-p}\,.$$

We still need to calculate the total heat contained in the whole lasagna and divide it by the total mass. The heat contained in one layer can be calculated as $Q_i = m_i T_i c = m_0 p^i T_0 q^i c = Q_0 p^i q^i$, where c is the specific heat capacity of the lasagna and $Q_0 = m_0 T_0 c$ is the heat in the zeroth layer. Therefore, we will get the total heat using the same procedure we used to get the total mass

$$Q = \frac{Q_0}{1-pq} = \frac{m_0 T_0 c}{1-pq}\,.$$

If the whole lasagna settles at the thermal equilibrium, the following must be true for its temperature

$$T_\mathrm{v} c M = Q$$
$$T_\mathrm{v} = \frac{\frac{m_0 T_0 c}{1-pq}}{c\frac{m_0}{1-p}}$$
$$T_\mathrm{v} = T_0 \frac{1-p}{1-pq} = 600\,\mathrm{K}\,.$$

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Problem CD ... lenses for nothing

Consider two lenses at a mutual distance $d = 25\,\mathrm{cm}$. The first is a biconvex lens with focal length $f = 10\,\mathrm{cm}$, and the second is a biconcave lens with focal length $-f$. How far from the first lens do we need to place the object for the system to project it on itself?

Jarda likes to project himself on himself.

Let us denote the object distance from the connecting lens as a. According to the Gaussian lens formula, the object will appear through the lens at a distance

$$a' = \frac{af}{a-f}\,.$$

The distance of this image from the second lens is $a_2 = d - a'$. We will use the Gaussian lens formula for the second time and get

$$a_2' = \frac{-a_2 f}{a_2 + f} = \frac{-\left(d - a'\right) f}{d - a' + f} = \frac{-\left(d - \frac{af}{a-f}\right) f}{d - \frac{af}{a-f} + f}\,.$$

According to the problem statement, the position of the image is equal to the position of the object, therefore $a_2' = -d - a$. So we are trying to solve the equation

$$\frac{-\left(da - df - af\right) f}{da - df - f^2} = -d - a\,,$$

from which we get

$$a^2 + \left(d - 2f\right) a - df = 0\,.$$

The solution to this equation is

$$a = \frac{-d + 2f \pm \sqrt{d^2 + 4f^2}}{2}\,.$$

Since we place the object in front of the first lens, the solution will be a root with a positive sign. The answer is $a = 13.5\,\mathrm{cm}$

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Problem CE ... heating with hydrogen

Electrolysis of water produces oxygen and hydrogen, the latter of which we capture and store in an inflatable balloon. The process takes place at a voltage of $1.48\,\mathrm{V}$, *electric current of* $15\,\mathrm{A}$ *and produces* $0.16\,\mathrm{g}$ *of hydrogen molecules* H_2. *Next, we will let the hydrogen burn in the air to heat a water container. The container has a heat capacity* $22\,\mathrm{J{\cdot}K^{-1}}$ *and an inner volume of* $55\,\mathrm{ml}$ *filled with water with initial temperature* $23\,\mathrm{^\circ C}$. *After the hydrogen gets burned, the temperature inside the container rises to* $94\,\mathrm{^\circ C}$. *What percentage of energy was used to heat the container and water?* *Jarda wanted to warm himself up with water.*

The molar mass of hydrogen approximately equals $M_{\mathrm{H_2}} = 2.0\,\mathrm{g{\cdot}mol^{-1}}$ because the hydrogen molecule is composed of two protons (as a matter of fact, it is $M_{\mathrm{H_2}} = 2.016\,\mathrm{g{\cdot}mol^{-1}}$, but in this case, less precision is sufficient). Mass $m = 0.16\,\mathrm{g}$ corresponds to the number of molecules by $n = N_\mathrm{A} m / M_{\mathrm{H_2}}$. The number of hydrogen atoms is double that of molecules, thus the total electric charge in the reactions is

$$Q = 2\frac{N_\mathrm{A} m}{M_{\mathrm{H_2}}} e\,.$$

The work done by the electrolyzer is simply

$$W = 2U\frac{N_\mathrm{A} m}{M_{\mathrm{H_2}}} e\,.$$

We got this energy, which stored itself in the hydrogen molecules by breaking up the water molecules. On the other hand, if we burn the hydrogen in the air, the energy will be released as heat.

By heating the container and the water, we have added heat

$$Q = \left(C + V_\mathrm{w} \rho_\mathrm{w} c_\mathrm{w}\right)\left(t_2 - t_1\right)\,,$$

where C is the heat capacity of the container, V_w, ρ_w and c_w are the volume, density and Specific heat capacity of the water in the container and $t_2 - t_1$ is the difference between the initial and final temperature.

The desired efficiency is

$$\eta = \frac{Q}{W} = \frac{M_{\mathrm{H}_2}\left(C + V_\mathrm{w}\rho_\mathrm{w}c_\mathrm{w}\right)(t_2 - t_1)}{2UeN_\mathrm{A}m} = 78\,\%\,.$$

Note that in order to heat this amount of water to nearly the boiling point, we used only 0.16 g of hydrogen! Hydrogen has the highest calorific value per mass unit of all chemical fuels.

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Problem CF ... a cart with a plumb bob

Let us consider a hill with a slope at an angle α. We place a wagon with a mass M on top of the hill. There is a string in the wagon of a length l hanging from the roof with a point mass m at its end (this mass is not included in the cart's mass). We then release the wagon down the hill. What angle relative to the vertical direction does the string stabilize at? Submit a positive result if the string is deflected in the direction of travel or negative if it is deflected in the opposite direction. The wagon moves without friction. *Lego has had this idea for a long time and he has no clue why he hadn't written this problem earlier.*

We are interested in the steady state situation, i.e., when the rope and the point mass stop moving with respect to the wagon. Then, they appear as if they formed one perfectly rigid body together with the wagon. We can calculate the acceleration of this body moving down the hill.

Its total mass is $M + m$, and the component of the gravitational force in the direction parallel to the hill is $(M + m)\,g\sin\alpha$. Thus the acceleration is $a = g\sin\alpha$.

Let us transfer to a system accelerating with the wagon. If the point mass suspended on the rope in this system is supposed not to move, there must be a zero resultant force. So now, we need to discuss the forces acting on it. The gravitational force mg acts vertically downward; the inertial force ma acts parallel to the roof toward the rear; and finally, there is a force exerted by the rope on which it hangs. The magnitude and direction of the force from the rope is (in a steady situation precisely such that this force compensates the two remaining forces. It is essential to realize that the direction of the force from the rope is in the same direction as the rope. Thus, we must find the direction of the vector sum of the remaining two forces.

The vertical component of the inertial force has magnitude $ma\sin\alpha = mg\sin^2\alpha$, and it is directed upward. The horizontal component has magnitude $ma\cos\alpha = mg\sin\alpha\cos\alpha$, which is directed backward. The combined force of gravity and inertia thus has a component in the vertical direction $mg\left(1 - \sin^2\alpha\right) = mg\cos^2\alpha$ pointing downward and a horizontal component $mg\sin\alpha\cos\alpha$ pointing backward. Note that for the limiting case of a vertical hill ($\alpha = \pi/2$), the mass point does not experience any force in the system associated with the wagon. This is due to the fact that in this situation, the system free falls, and the point mass is in a weightless state from the viewpoint of this system.

Now back to the angle of deflection of the rope. We are interested in its deflection with respect to the vertical direction. We obtain this angle as the inverse tangent of the ratio of the horizontal and the vertical component

$$\beta = \arctan\frac{mg\sin\alpha\cos\alpha}{mg\cos^2\alpha} = \arctan\frac{\sin\alpha}{\cos\alpha} = \arctan(\tan\alpha) = \alpha\,.$$

The string will be deflected backward at calculated angle α; therefore, the desired result is $-\alpha$.

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Problem CG ... hot air balloon

A hot air balloon with volume $V = 3\,000\,\mathrm{m^3}$ and mass $M = 724\,\mathrm{kg}$ (without air) ascends to the sky. The temperature of the air inside the balloon is $T_\mathrm{b} = 120\,\mathrm{^\circ C}$, while the temperature of the surrounding air is constant $T_0 = 20\,\mathrm{^\circ C}$. However, as the balloon goes up, the ambient pressure drops. Determine at what value of ambient pressure does the weight of the balloon equal the buoyant force acting on it? The volume of the rest of the balloon is negligible compared to the volume of air in it. *Lego recalled the Physics Olympiad.*

We know that at temperature T_0 and pressure p_0 the density of air is ρ_0. The quantities in the equation of state imply, that the density is proportional to N/V, hence we can write

$$pV = NkT$$
$$\frac{p}{kT} = \frac{N}{V} \sim \rho$$
$$\frac{p}{T}C = \rho\,,$$

where C is a constant whose value can be determined from the density under normal conditions.

$$C = \frac{T_0\rho_0}{p_0}$$

By back-substitution, we get the air density under general temperature and pressure.

$$\rho = \rho_0\frac{T_0 p}{T p_0}\,.$$

Thus the mass of air in the balloon is at pressure p

$$m = V\rho_0\frac{T_0 p}{T_\mathrm{b} p_0}\,,$$

and the mass of air displaced by the balloon

$$m_\mathrm{vyt} = V\rho_0\frac{p}{p_0}\,.$$

The task asks at what pressure p the gravitational and buoyant forces are balanced

$$F_\mathrm{g} = F_\mathrm{vz}$$
$$(M+m)g = m_\mathrm{vyt}g$$
$$M + V\rho_0\frac{T_0 p}{T_\mathrm{b} p_0} = V\rho_0\frac{p}{p_0}$$
$$M = V\rho_0\frac{p}{p_0}\left(1-\frac{T_0}{T_\mathrm{b}}\right)$$
$$p = p_0\frac{1}{1-\frac{T_0}{T_b}}\frac{M}{V\rho_0} = 80\,\mathrm{kPa}\,.$$

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Problem CH ... heating

In scientific applications, it is sometimes necessary to heat a sample suspended in a vacuum. This is done using a stream of accelerated electrons, whose kinetic energy is converted to thermal energy upon impact. Consider a cathode with an emission stream of electrons I_e, which are accelerated by a voltage U. These electrons hit a sample with a total surface area S. Assume that the surface is small enough that the temperature is the same everywhere on it, and all of the electron energy is converted into heat. At what temperature T, will the sample stabilize? Consider that it behaves like a perfect black body, i.e., heat loss is only in the form of radiation, the ambient temperature is much lower than T, and neglect the initial velocity of the electrons upon emission.

The instrument measures so often that some elementary functions are memorized at last...

We assume a constant emission current I_e, which is defined as usual

$$I_e = \frac{Q}{t}\,,$$

where Q is the charge of electrons transferred in time t.

Using the equation for the energy of the charge accelerated by the voltage U, we can express the power P heating the sample

$$E = QU = UI_e t \quad \Rightarrow \quad P = UI_e\,.$$

The magnitude of the power radiated to the surroundings from our sample is governed by the Stefan-Boltzmann law

$$P_{ok} = \sigma S T^4\,.$$

The temperature of the sample stabilizes when $P = P_{ok}$, thus we have

$$UI_e = \sigma S T^4 \quad \Rightarrow \quad T = \sqrt[4]{\frac{UI_e}{\sigma S}}\,.$$

According to the problem statement, we could neglect the incoming power of thermal radiation from the surrounding environment. Therefore, the last equation gives us the desired result.

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Problem DA ... cutting an apple

We want to cut a perfectly spherical apple with a radius $R = 3.6\,\mathrm{cm}$ so that only the core is left. At first, we cut off a section of the apple with a straight cut at a distance $a = 0.7\,\mathrm{cm}$ from the center of the apple so that the plane of the cut is parallel to the core's axis of symmetry. Perpendicularly to this plane, we make two other cuts at a distance a from the center. Again at a distance a from the center, we make the last cut parallel with the first one such that the core is a square with a side $2a$ when viewed from above. Assume that the force needed to make the cut is proportional to the length of the knife in the apple. Determine the ratio of the work done during the last cut and the first cut.

Jarda was sitting under a tree, and an apple fell on his head.

According to the problem statement, the force is proportional to the length of the cut, which we denote by l. We can write

$$F = kl\,,$$

where k is a constant of proportionality. If we make the cut $\mathrm{d}x$ deeper, we do the work $\mathrm{d}W$. When cutting through the entire apple, we perform the work

$$W = \int kl \,\mathrm{d}x = \int k \,\mathrm{d}S = kS\,,$$

so the work is directly proportional to the area of the cut.

The area of the first cut is in the shape of a circle. Its radius is according to the Pythagorean theorem $r = \sqrt{R^2 - a^2}$. Thus, the area of the first cut is $S_1 = \pi\left(R^2 - a^2\right)$.

The plane of the last cut is also at a distance a from the center and the shape of the cut is a subset of a circle, whose radius is also r. A strip of width $2a$ is cut out from this circle, and we need to calculate its area.

The chord intersecting the circle has a length $2t = 2\sqrt{r^2 - a^2}$ according to the Pythagorean theorem. From the circle's center, we see it at an angle $2\alpha = 2\arctan(t/a)$. The area of the circular sector with this central angle is $S_\mathrm{v} = (2\alpha/2\pi)\pi r^2$.

Now, let us calculate the area of the triangle ABS. It is $S_\mathrm{t} = ta$. The area of the segment is, therefore, $S_\mathrm{u} = S_\mathrm{v} - S_\mathrm{t} = \alpha r^2 - ta$. We subtract the area of this segment two times (for each side) from the area of the circle with a radius r, so the area of the last cut is

$$S_2 = (\pi - 2\alpha)\, r^2 + 2ta = \left(\pi - 2\arctan\left(\frac{\sqrt{R^2 - 2a^2}}{a}\right)\right)\left(R^2 - a^2\right) + 2a\sqrt{R^2 - 2a^2}\,.$$

The ratio of work we are looking for is equal to the ratio of the areas, which is

$$\frac{S_2}{S_1} = \frac{\left(\pi - 2\arctan\left(\frac{\sqrt{R^2-2a^2}}{a}\right)\right)\left(R^2 - a^2\right) + 2a\sqrt{R^2 - 2a^2}}{\pi\left(R^2 - a^2\right)} \doteq 0.251\,.$$

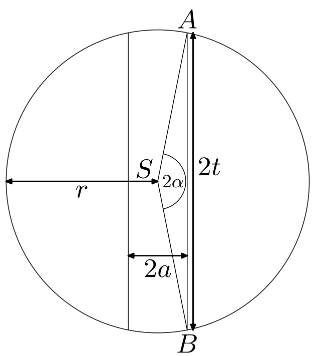

Fig. 1: The cut of the apple.

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Problem DB ... emergency and problematic

"Fireman" jumps onto the pole, but suddenly the pole breaks off the ground, and he begins reassessing the situation. He will either fall with the pole to the floor, holding onto the top of the rod throughout the whole fall. Or he may let go of the bar and free fall from the initial height. What is the ratio of the impact velocities from the first and second situations? The length of the pole is l, its mass is M, and the mass of the fireman is m.

Consider zero initial velocity in both cases, and the pole rotates about an axis passing through the broken end just off the ground. The size of the "fireman" is negligible compared to the length of the pole. *Pepa wanted to call "fireman".*

Let us look at the first case, where the fireman holds onto the pole and falls with it. In this case, the whole system is in rotational motion around the base of the pole. We will calculate the impact velocity using the law of conservation of energy. In the beginning, the rod is perpendicular to the ground, so its center of gravity is at height $l/2$; and the firefighter's center of gravity is at height l. The potential energy of the whole system in the beginning is

$$E_0 = mgl + Mg\frac{l}{2}\,.$$

At the end of the fall, the total kinetic energy of the system is equal to

$$E = \frac{1}{2}I\omega^2\,,$$

where I is the moment of inertia of the entire system. We can calculate it as the sum of moments of inertia, both the fireman: $I_h = ml^2$ and the pole, which in case of rotation about one end equals $I_t = Ml^2/3$. Thus we get the equation

$$mgl + Mg\frac{l}{2} = \frac{1}{2}\left(ml^2 + \frac{1}{3}Ml^2\right)\omega^2\,.$$

Now, we can express the fall velocity of the fireman $v_1 = \omega l$:

$$2\frac{mgl + Mg\frac{l}{2}}{m + \frac{1}{3}M} = v_1^2\,,$$

$$v_1^2 = 2gl\frac{m + \frac{M}{2}}{m + \frac{M}{3}}\,,$$

$$v_1 = \sqrt{2gl\frac{m + \frac{M}{2}}{m + \frac{M}{3}}}\,.$$

In the second case, the fireman will let go of the rod immediately, i.e., he will experience free fall. We can once again use the law of conservation of energy. In the beginning, he has only potential energy mgh; and in the end, only kinetic energy

$$mgl = \frac{1}{2}mv_2^2\,,$$

$$v_2^2 = 2gl\,,$$

$$v_2 = \sqrt{2gl}\,.$$

So the ratio of the fall velocities in both cases is

$$\frac{v_1}{v_2} = \sqrt{\frac{m + \frac{M}{2}}{m + \frac{M}{3}}}\,.$$

Note that for a non-zero mass of the rod, we get that the fireman holding onto the pole will fall with greater impact velocity compared to the case when he lets go of it immediately.

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Problem DC ... floating

Let us have a buoy made of a hollow sphere of outer radius $r = 51\,\mathrm{cm}$ and thickness $t = 1\,\mathrm{cm}$ whose material has density $\rho = 854\,\mathrm{kg{\cdot}m^{-3}}$. A rod of linear density $\lambda = 0.74\,\mathrm{kg{\cdot}m^{-1}}$ is attached to it on the outside perpendicular to the sphere's surface. In the buoy, there is a weight of mass $m_\mathrm{z} = 1.12\,\mathrm{kg}$ (of negligible dimensions), which is fixed inside the sphere on the opposite side as the rod.

The buoy floats on the water; the rod is initially in a vertical position above the sphere. What can be the maximum length of the rod l so that the buoy is stable, i.e. the buoy returns to its original position after a small deflection? *Martin and the mechanics exam.*

The position where the rod is vertically above or under the buoy is the equilibrium position. That follows from symmetry. However, one of these vertical positions is stable, while the other is unstable. A stable equilibrium is one, where there is (locally) a minimum of potential energy. In an unstable equilibrium, there is a maximum of potential energy. We need to express the potential energy as a function of the deflection of the rod with respect to the vertical direction above the buoy.

The mass of the whole buoy is constant, so the same volume of the buoy will be submerged at all times. Moreover, the rod will not go under the water when deflected by a bit, so the submerged part will always have the shape of a spherical cap of radius r and with a given volume. Therefore, even though the hollow sphere may shift a bit in the horizontal direction by tilting the buoy in the vertical direction, the sphere will remain at the same height. That implies two things – the potential energy of the sphere itself does not change because its center (which is also the center of gravity) does not change its height; the potential energy of the water in which the buoy floats does not change either.

It remains for us to calculate how the potential energy of the weight and the rod changes. The weight is fixed on the inside of the sphere at a distance $r - t$ from the sphere's center. If we put the zero point of the height to the center of the sphere, the potential energy of this weight will be $m_\mathrm{z} g(r - t)(-\cos\varphi)$, where φ is the angle that the rod makes with the vertically upward direction.

The mass of the rod will be $m_\mathrm{t} = l\lambda$, and the distance and its center of gravity will be in the center of the rod. The center of gravity will be at a distance $r + l/2$ from the center of the sphere, and its potential energy will be $l\lambda g(r + l/2)\cos\varphi$. The sum of the potential energy of the weight and the rod will be

$$E_\mathrm{p}(\varphi) = g(l\lambda(r + l/2) - m_\mathrm{z}(r - t))\cos\varphi\,.$$

The cosine has a local maximum in $\varphi = 0$; if we want a local minimum for the position when the rod direction is vertically upwards, the expression in the brackets must be negative. We get a condition

$$0 > \frac{\lambda}{2} l^2 + r\lambda l - m_z (r - t) \,.$$

The discriminant is $D = r^2\lambda^2 + 2\lambda m_z(r - t)$, therefore the roots are

$$l_{1,2} = \frac{-r\lambda \pm \sqrt{r^2\lambda^2 + 2\lambda m_z(r - t)}}{\lambda} \,,$$

where the negative root does not interest us and looking at the initial inequality, we see that the positive root gives us an upper bound for the length of the rod at which the vertically upward position will still be stable. This maximum length is

$$l_{\max} = -r + \sqrt{r^2 + 2\frac{m_z(r - t)}{\lambda}} = 82\,\mathrm{cm} \,.$$

Notes: Looking again at the inequality we derived, we see that this condition is equivalent to the condition that the resulting center of gravity is below the center of the sphere. That makes sense because when the buoy rotates, the center of gravity goes up, and its potential energy therefore increases.

How would we solve this through the forces? The important points would again be the center of gravity of the buoy and the center of buoyancy, which is the point at which we could place the center of buoyant force. This point is obviously (by symmetry) somewhere below the center of the sphere. So, when we deflect the buoy, the center of gravity and the center of buoyancy will not be directly under each other, and thus the forces acting in the two centers there will be spinning the buoy with their torque. In case the center of gravity is lower than the center of the sphere, this torque will counteract the deflection, and therefore it will be in a stable position and vice versa.

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Problem DD ... snowfall

Overnight, 6 cm *of snow fell, so in the morning, Jarda went to shovel it out of the sidewalk. He took a big shovel with a width of* $d = 60\,\mathrm{cm}$, *put it on the ground, and started pushing it in front of him with a velocity of* $v = 0.6\,\mathrm{m\cdot s^{-1}}$. *In this process, the snow with a density of* $\rho = 120\,\mathrm{kg\cdot m^{-3}}$ *accumulates on the shovel. The coefficient of friction between the shovel and the ground is* $f = 0.6$; *do not account for the shovel's mass. How long will it take Jarda to stop if he can exert a maximum force of* $F = 60\,\mathrm{N}$? *Consider that snow on the shovel moves with the shovel.* *Jarda's greenhouse is covered in snow…*

The snow shovel is subject to two forces. The first is a frictional force of magnitude $F_\mathrm{t} = fF_\mathrm{N} = fmg$, where m is the mass of snow on the shovel and g is the gravitational acceleration. In addition, Jarda must accelerate the accumulated snow to a velocity v, which requires another part of the force F_p. The mass m of the accumulated snow increases with time as $m = \rho h d v t$, so the corresponding force is

$$F_\mathrm{p} = \frac{\mathrm{d}m}{\mathrm{d}t} v = \rho d h v^2 \,,$$

where h is the height of the snowfall, d is the width of the shovel. We see that the force is constant over time.

Thus, the total force exerted by the Jarda is

$$F_\mathrm{p} + F_\mathrm{t} = \rho d h v \left(v + t g f \right) \ .$$

Putting $F = F_\mathrm{p} + F_\mathrm{t}$ gives us the time

$$t = \frac{F}{\rho d h v g f} - \frac{v}{g f} = 3.8\,\mathrm{s}\,.$$

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Problem DE ... USA annexes

The border between Canada and the USA is said to contain the longest straight line, which length is $l = 2\,057\,\mathrm{km}$. However, this is not true because this part of the border runs along the forty-ninth circle of latitude. Calculate by how many kilometers the length of this "straight" part of the border would be reduced if it were a truly straight line (i.e., the shortest line) running along the surface of a sphere whose endpoints remained the same.

Matěj is fascinated by strange borders `https://www.youtube.com/shorts/caJeL1sjqJQ`.

It is not a straight line because the circles of latitude (except the equator) are not straight. If we wanted to walk along a circle of latitude, we would have to keep turning slightly. The forty-ninth circle of latitude is a circle with radius $r = R_\oplus \cos 49\,^\circ$. The current border between Canada and the USA is a circular arc with an angle of $\alpha = l/r$. The actual distance of the endpoints in 3D space is $d = 2r\sin(\alpha/2)$.

A straight line running along the surface of a sphere is always a circle centered at the center of the sphere and with radius $R_\oplus$. This is because it is the shortest path on the surface that connects the two points. To calculate the magnitude of the angle β of the arc of a truly straight border, we use the same formula as in the paragraph above, but in the inverse form $\beta = 2\arcsin\left(d/(2R_\oplus)\right)$. The length of this arc is, therefore, $\beta R_\oplus$. Putting it together, we get

$$\Delta l = l - 2R_\oplus \arcsin\left(\cos 49^\circ \sin \frac{l}{2R_\oplus \cos 49^\circ}\right) = 12\,\mathrm{km}\,.$$

Note: the notion of a straight line on a curved surface (e.g., the surface of a sphere) has a precise definition in differential geometry. These lines are called geodesics and have a special meaning in general relativity. Here, geodesics describe the trajectories of objects in 4D space that is curved by the presence of physical bodies (to give an idea, e.g., the motion of things falling into a black hole).

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Problem DF ... wind velocity

On a warm summer afternoon, Robo hides from a storm under a shelter where he stands at a distance $R = 1.2\,\mathrm{m}$ from the edge. At what velocity is the wind blowing against him if it is pouring down on him from his feet to his shoulders? The shelter's roof is at height $H = 2.3\,\mathrm{m}$, and Robo's shoulders are at height $h = 1.5\,\mathrm{m}$. Assume that the wind blows only horizontally and that the droplets are subject to air resistance (drag). Assume the droplets to be small spheres with radius $r = 0.8\,\mathrm{mm}$. The drag coefficient for a sphere is $C = 0.5$.

Rain is fine if you have a place to hide.

First, let us note that droplets move at a constant velocity. In the vertical direction, air resistance acts on them, and in the horizontal direction, they are carried by the wind.

When we draw a picture, we see that we get a right triangle with legs R and $H - h$ and a hypotenuse that characterizes the trajectory of the droplet. We'll denote the angle between the hypotenuse and leg R θ and note that the same angle is formed by the velocity components v_x in the horizontal direction and v in the direction of the droplet's trajectory. Then

$$\tan\theta = \frac{H-h}{R} = \frac{v_y}{v_x}\,,$$

where v_y is the steady velocity component in a vertical direction. This velocity is determined from the equilibrium of the gravitational force F_g of the droplet and the drag force F_o. Thus

$$\begin{aligned} F_g &= F_o\,,\\ mg &= \frac{1}{2}CS\rho v_y^2\,,\\ \frac{4}{3}\pi r^3\rho_\mathrm{w} g &= \frac{1}{2}C\pi r^2\rho v_y^2\,,\\ v_y &= \sqrt{\frac{8g\rho_\mathrm{w} r}{3C\rho}}\,, \end{aligned}$$

where we used the spherical shape of the droplet and its cross-section $S = \pi r^2$. The density of water is ρ_w, and the air density is ρ. We noted that the droplet's component v_x is purely made up of the wind velocity, and so we get a final expression for the wind velocity

$$v_x = \frac{R}{H-h}\sqrt{\frac{8g\rho_\mathrm{w} r}{3C\rho}} = 8.9\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem DG ... race of shadows

Legolas was walking at night along a sidewalk parallel to street lights, observing the movement of his shadows on the ground. The lights, however, were of varying heights, so the shadows moved strangely. Lego wants to calculate the speed at which the two shadows cast by his head move relative to one another. Assume Legolas' head is a point at height $h_\mathrm{h} = 1.7\,\mathrm{m}$, moving at speed $v = 1.5\,\mathrm{m{\cdot}s^{-1}}$. Lamps are placed on a straight line that is $d = 2.1\,\mathrm{m}$ away from the line that Lego is walking on, and the bases of the lamps are $l = 13\,\mathrm{m}$ apart. Their heights *are* $h_1 = 3.1\,\mathrm{m}$ *and* $h_2 = 3.6\,\mathrm{m}$ *and think of them as point sources. Legolas is interested in the magnitude of the relative speed of the shadows when he is separated from the bases of both lamps equally.* *Legolas was walking down the sidewalk.*

The triangle given by the head's shadow, Lego's head, and Lego's feet is similar to the triangle given by the head's shadow, the base of the lamp, and the point source itself. Hence their angles are identical.

If we project the whole situation into a plane perpendicular to Lego's direction, we see Lego's projection will not move at all; namely, Lego's feet will be away from the base of the lamp by d. Thus, the shadow in this projection will not move either and will be at distance $t_{1,2}$ from the lamp's base, where (from the similarity triangles)

$$\frac{t}{h_{1,2}} = \frac{t-d}{h_\mathrm{h}}\,,$$
$$d\frac{h_{1,2}}{h_{1,2}-h_\mathrm{h}} = t_{1,2}\,.$$

Thus, the velocities of the shadows are parallel to the direction in which Legolas is walking. Since the velocity of one shadow with respect to the other is given by the difference between these two vectors, we see that it is sufficient to subtract their magnitudes.

So what are those magnitudes? Again we use the similarity of triangles, this time as seen from the top view. We remember Lego's original position and the original position of the shadow and let Lego move. Then the triangle formed by the lamp and the Lego's original and new position is similar to the triangle formed by the lamp and the original and new position of the shadow. Since we also know that the ratio of the distance between the shadow and the lamp to the distance of the Lego and the lamp is $t_{1,2}/d$, we know that the ratio of shadow displacement to Lego's displacement is the same. Thus, the shadow of Lego's head moves $t_{1,2}/d$-times faster than Lego's, that is $v_{1,2} = vt_{1,2}/d$.

It only remains, to plug in the numbers and subtract these two magnitudes of velocity

$$\Delta v = v_1 - v_2 = \frac{v}{d}\left(t_1 - t_2\right) = \,,$$
$$\Delta v = \frac{v}{d}\left(d\frac{h_1}{h_1-h_\mathrm{h}} - d\frac{h_2}{h_2-h_\mathrm{h}}\right)\,,$$
$$\Delta v = v\frac{h_\mathrm{h}(h_2-h_1)}{(h_2-h_\mathrm{h})(h_1-h_\mathrm{h})}\,,$$
$$\Delta v = 0.48\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem DH ... termoanemometer

To measure the speed of water flow in a tube, we can use a wire, which we place into the tube. We run a constant current $I = 200\,\mathrm{mA}$ *through the wire and measure the voltage on it. For volumetric flow* $Q = 70\,\mathrm{ml{\cdot}s^{-1}}$ *we measured the voltage* $U = 522\,\mathrm{mV}$*. We then increased the flow rate by* $\Delta Q = 8\,\mathrm{ml{\cdot}s^{-1}}$ *while keeping the same current, and the voltage dropped by* $\Delta U = 25\,\mathrm{mV}$ *due to the temperature change of the wire resistance. How did the average temperature* *of the fluid change after passing through the device compared to the first measurement? Also state whether the temperature is now higher or lower.*

Jarda has heard of a device with a strange name.

Let us denote the temperature of the water before entering the device by T_0. In the first case, the temperature of the water after exiting the device is

$$T_1 = T_0 + \frac{IU}{Q\rho c}\,,$$

where c is specific heat capacity of water and ρ its density. In the second case, it holds

$$T_2 = T_0 + \frac{I\,(U - \Delta U)}{(Q + \Delta Q)\,\rho c}\,.$$

We then subtract these two equations and obtain the result

$$T_2 - T_1 = \frac{I}{\rho c}\left(\frac{U - \Delta U}{Q + \Delta Q} - \frac{U}{Q}\right) = -5.2 \cdot 10^{-5}\,\mathrm{K}\,.$$

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Problem EA ... clumsy Danka

Danka dropped a small ball with a mass $m = 10\,\mathrm{g}$ from a height $h = 1.3\,\mathrm{m}$ above the floor so that it landed on the "bosu" at a horizontal distance $x_0 = 5\,\mathrm{cm}$ from its axis of symmetry and bounced off it elastically. How far from the axis of the bosu did the ball fall after the bounce? The bosu is a rubber exercise mat shaped like a hemisphere with a radius of the base equal to 29 cm. Assume that the bosu does not move on impact. *Danka's brother owns bosu.*

The ball's movement consists of a free fall, a perfectly elastic bounce from the bosu, and a parabolic projectile motion. Let's look at them one at a time. The height at which the ball bounces off the bosu indicates how much of its potential energy is converted into kinetic energy, giving us the initial velocity of the projectile motion. We shall introduce a coordinate system with the origin at the center of the base of the bosu. The x-coordinate defines the horizontal distance from the center of the bosu, and the y-coordinate determines the vertical distance. Let us denote by y_0 the height at which the ball bounces from the bosu. We can calculate it by applying Pythagorean theorem to the triangle in the figure as

$$y_0 = \sqrt{r^2 - x^2} \doteq 28.565\,7\,\mathrm{cm}\,.$$

Using the law of conservation of energy, we then calculate the velocity v_0 of the ball after bouncing as

$$mg(h - y_0) = \frac{1}{2}mv_0^2\,,$$

$$v_0 = \sqrt{2g(h - y_0)} \doteq 4.461\,1\,\mathrm{m{\cdot}s^{-1}}\,.$$

Next, we need to calculate the angle of the ball's velocity after bouncing off the bosu with respect to the horizontal plane. Let us denote the angle by θ. When we consider a perfectly elastic collision, the angles of incidence and ricochet are identical. We see in the figure that

$$90^\circ = 2\alpha + \theta\,.$$

We calculate the angle α using the aforementioned triangle as

$$\alpha = \arcsin\frac{x}{r}\,.$$

We get

$$\theta = 90^\circ - 2\arcsin\frac{x}{r} \doteq 70.14^\circ$$

Then we compute the components v_{0x} and v_{0y} of the velocity at the beginning of the projectile motion as

$$v_{0x} = v_0\cos\theta\,,\qquad v_{0y} = v_0\sin\theta\,.$$

Now we know both the initial coordinates and the initial velocity components. We can therefore write the equations of motion

$$x = x_0 + v_{0x}t\,,\qquad y = y_0 + v_{0y}t - \frac{1}{2}gt^2\,.$$

Time t starts when the ball bounces off the bosu. When the ball hits the ground, its y-coordinate is zero. From there, we get the quadratic equation

$$0 = y_0 + v_0\sin\theta t - \frac{1}{2}gt^2\,,$$

and consequently, the variable t

$$t = \frac{v_0\sin\theta \pm \sqrt{(v_0\sin\theta)^2 + 2gy_0}}{g} \doteq 0.918\,8\,\mathrm{s}\,.$$

Since we are looking for a positive value of time, the only real solution is the one with a positive sign in front of the square root. The final step is to plug this expression into the equation for x, and get the numerical result.

Ideally, the resulting equation for x should contain only the quantities defined in the problem statement. In this case, however, the relationship would be too intricate, and it is very easy to make a mistake when substituting numerical values in. Therefore, it is better to calculate the partial results of the crucial variables with sufficient precision, as we have indeed done while solving this problem, and plug these into the final formula. We arrive at the result that the ball hits the ground at distance $x = 1.44\,\mathrm{m}$ from the bosu's axis.

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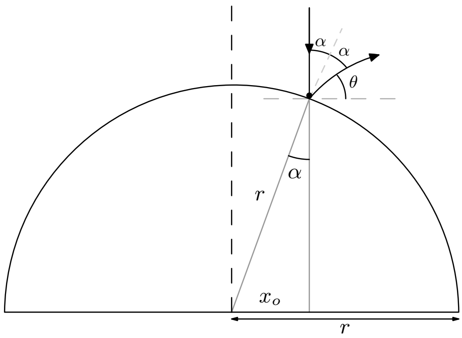
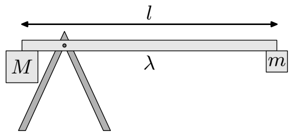


Fig. 2: A sketch of the situation.

Problem EB ... simplified trebuchet



The arm of a trebuchet has a length of $l = 9.14\,\mathrm{m}$, with the axis of rotation dividing it in a ratio of 1 : 5. The linear density of the arm is $\lambda = 10\,\mathrm{kg{\cdot}m^{-1}}$. For simplicity, assume that the weight and the projectile are mass points attached at the ends of the arm. The weight has a mass of $M = 15\,\mathrm{t}$ and is attached at the shorter end. The projectile has a mass $m = 60\,\mathrm{kg}$ and is attached at the longer end. What is the angular acceleration when the arm is horizontal?

Legolas does not need just pulleys...

Probably the most straightforward way to get the angular acceleration is to divide the total torque of the system by the total moment of inertia, all of which must be calculated with respect to the axis of rotation. Since both quantities are additive, we can calculate them by summing contributions from the weight, projectile, and arm.

The torque of the weight is simply $M_z = Mgl/6$. Similarly, the torque of the projectile is $M_\mathrm{p} = -5mlg/6$, where we used the minus sign to reflect that this torque tends to rotate the arm in the opposite direction as the weight. The torque of the arm itself also acts in the opposite direction. Its mass is $l\lambda$ and its center of gravity is at a distance $2l/6 = l/3$ from the axis of rotation. The torque of the arm is, therefore $M_\mathrm{r} = -g\lambda l^2/3$. The resulting torque acting on the arm with respect to the axis of rotation is

$$M_\mathrm{v} = M_\mathrm{z} + M_\mathrm{p} + M_\mathrm{r} = gl\left(\frac{M}{6} - \frac{5m}{6} - \frac{l\lambda}{3}\right).$$

Moment of inertia of a point mass m, with respect to the rotational axis at distance r is mr^2, thus the moment of inertia of the weight is $J_z = Ml^2/36$ and the moment of inertia for the projectile is $J_p = 25ml^2/36$.

The moment of inertia of the arm about its center is $ml^2/12$. But we need the moment of inertia about the axis of rotation. We can get this using Steiner's theorem, i.e., by adding mr^2, where r is the distance between the center of gravity of the arm and the axis of rotation, in our case $l/3$. All in all, we get

$$J_r = J_0 + J_S = \frac{1}{12}\lambda l^3 + \frac{1}{9}\lambda l^3 = \frac{7}{36}\lambda l^3 .$$

The resulting moment of inertia is therefore calculated as

$$J = J_\mathrm{z} + J_\mathrm{p} + J_\mathrm{r} = \left(\frac{M}{36} + \frac{25m}{36} + \frac{7}{36}\lambda l\right) l^2 \,.$$

Now, we just need to substitute obtained results into the formula for angular acceleration

$$\varepsilon = \frac{M_\mathrm{v}}{J} = \frac{gl\left(\frac{M}{6} - \frac{5m}{6} - \frac{l\lambda}{3}\right)}{\left(\frac{M}{36} + \frac{25m}{36} + \frac{7}{36}\lambda l\right) l^2}$$

$$\varepsilon = 6\frac{g}{l}\frac{M - 5m - 2l\lambda}{M + 25m + 7\lambda l} = 5.5\,\mathrm{s^{-2}} \,.$$

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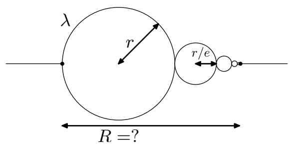
Problem EC ... repulsive 2D snowman

We have a conductive ring with linear resistance λ and radius r. We also have a ring of the same material but with a radius e-times smaller. Surprisingly, we gradually found infinitely many more such rings, each with a radius of $1/\mathrm{e}$ compared to the previous one. We arrange them side by side in the shape of a snowman and connect them conductively. What will be the total resistance of this shape between the endpoints on the axis of symmetry? *Do not disturb Jarda's circles.*



First, we determine the ratio between the two opposite sides of a circle of radius R. These are actually two resistors in parallel, each with a resistance $\pi r\lambda$. The total resistance of the ring is thus $R_0 = \pi r\lambda/2$.

This relationship holds for each of the rings. Once we connect them conductively in series, their total resistance is given by the simple sum. The only thing that changes for each ring is its radius. We get

$$R_\mathrm{c} = \frac{\pi r\lambda}{2}\sum_{n=0}^{\infty}\frac{1}{\mathrm{e}^n} = \frac{\pi r\lambda}{2}\frac{\mathrm{e}}{\mathrm{e}-1} \,,$$

where we used the formula for the sum of an infinite series.

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Problem ED ... searching for a type II civilization

Organizers of FYKOS are flying through space, looking for new civilizations to where they could expand. So far, they have found only one ring that could be a remnant of a Dyson sphere. It has a radius of $R = 1.2 \cdot 10^8$ m, but the organizers are also interested in its mass. Therefore, they flew to the ring's center with their rocket and then moved perpendicularly to its cross-section. When they waited long enough, they discovered that their oscillatory period was $T = 60$ h. What is the total ring's mass if we assume that the mass is uniformly distributed and the amplitude of the oscillations is an order of magnitude smaller than R? *Pepa likes to swing.*

Firstly, we will calculate the force F exerted on the organizers when they are displaced by a small distance z in the direction of the ring's axis. A ring's element of mass $\mathrm{d}M$ will then exert a force with a magnitude $\mathrm{d}F = Gm\,\mathrm{d}M/(R^2+z^2)$ on the organizers. When summing these forces, the components in the ring's cross-section cancel out. Only the components perpendicular to the plane of the ring will remain, and for them, we can write

$$\frac{\mathrm{d}F_z}{\mathrm{d}F} = \sin\varphi = \frac{z}{\sqrt{(R^2+z^2)}}\,,$$

where φ is the angle, which is given by the line connecting the organizers to the element $\mathrm{d}M$ and the ring's cross-section. For $\mathrm{d}F_z$ the following holds

$$\mathrm{d}F_z = Gm\,\mathrm{d}M\frac{z}{(R^2+z^2)^{\frac{3}{2}}}\,.$$

In order to get the total force in the perpendicular direction to the ring's cross-section, we need to sum all these infinitesimal contributions. But notice that the force component $\mathrm{d}F$ is the same for all elements $\mathrm{d}M$. The sum(integral) is then drastically simplified, and we get the total force as

$$F_z = GmM\frac{z}{(R^2+z^2)^{\frac{3}{2}}}\,.$$

Now recall that the displacement z is small so we can neglect its second power in the denominator, and hence $F_z = GmMz/R^3$. The stiffness therefore is

$$k = \frac{GmM}{R^3}\,.$$

The final thing we need to do is to plug this formula into the equation for the period of a linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}}\,,$$

from where we can express the total mass of the ring as

$$M = 4\pi^2\frac{R^3}{GT^2} = 2.2\cdot 10^{25}\,\mathrm{kg}\,.$$

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Problem EE ... Doppler on a stroll

Christian Doppler took a stroll. After a while, he noticed that in both directions (in his direction and the opposite direction), people are walking at velocity $v = 1.0\,\mathrm{m\cdot s^{-1}}$ and with $l = 4.0\,\mathrm{m}$ spaces between them. He decided to take advantage of this to find the speed at which he must move to meet as few people as possible. How fast should he walk to meet the least number of people in time $T \gg l/v$? Find all solutions. *Legolas is very neuroatypical.*

Let us denote v_D the speed of Doppler. The time between the meeting of the two bypassers will be

$$t_\mathrm{p} = \frac{l}{v+v_\mathrm{D}}\,.$$

Therefore, the frequency of the meeting of the bypassing people is

$$f_\mathrm{p} = \frac{1}{t_\mathrm{p}} = \frac{v + v_\mathrm{D}}{l} \,.$$

Similarly, the frequency of meeting people going in the same direction will be

$$f_\mathrm{r} = \frac{1}{t_\mathrm{r}} = \frac{|v - v_\mathrm{D}|}{l} \,.$$

The absolute value then divides the solution into two cases where the passers-by bypass[2] Doppler ($v_\mathrm{D} \leq v$), and the case where the Doppler bypasses the passers-by ($v_\mathrm{D} \geq v$).

The number of people the Doppler meets is the product of the total frequency of meetings and the time T. Total frequency means the sum of the frequencies of meeting people in one direction and the other. Actually, it is more like an average frequency, but since we only need the total number of people Doppler meets and we know he meets a lot of people (because $T \gg l/v$), this figure is perfectly enough.

To get rid of the absolute value, let us split the problem into the cases discussed above.

Passer-by bypass Doppler ($v_\mathrm{D} \leq v$)

The final frequency will be

$$f_\mathrm{v1} = \frac{v + v_\mathrm{D}}{l} + \frac{v - v_\mathrm{D}}{l} = 2\frac{v}{l} \,,$$

which is a constant independent of the speed of Doppler. Therefore, the total number of people he meets will be

$$N_1 = f_\mathrm{v1} T = 2\frac{v}{l} T \,.$$

Doppler bypasses the passers-by ($v_\mathrm{D} \geq v$)

The final frequency will be

$$f_\mathrm{v2} = \frac{v + v_\mathrm{D}}{l} + \frac{v_\mathrm{D} - v}{l} = 2\frac{v_\mathrm{D}}{l} \,.$$

Therefore, the total number of people he meets will be

$$N_2 = f_\mathrm{v2} T = 2\frac{v_\mathrm{D}}{l} T \,,$$

so if the speed is greater than v Doppler will always meet more than N_1 people. Thus, the minimum number of people he will meet in a given time is N_1, and he will meet that many if he walks at a speed less than or equal to v, so our solution (since we had to find all the solutions) is $[0, v]$ or $[-v, v]$ if we assume he can also go in the opposite direction.

It is worth noting that if we interpret the crowd of passers-by as a wave with wavelength l and wave speed v, i.e. with frequency $f_0 = v/l$, we get by expanding the expression for the frequency of meeting

$$f_\mathrm{p} = \frac{v + v_\mathrm{D}}{l}\frac{v}{v} = \frac{v + v_\mathrm{D}}{v} f_0 \,,$$

which is the relation for the Doppler effect with a moving observer, as well as by analogy for f_r.

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[2] even if they travel at the speed

Problem EF ... pigeon doesn't fall far from the stone

A pigeon with a mass $M = 300\,\mathrm{g}$ is flying $h = 30\,\mathrm{m}$ above the ground at velocity $V = 20\,\mathrm{km\cdot h^{-1}}$. Suddenly, he is hit from behind (i.e., in the direction of his velocity) by a stone thrown from the ground with a mass $m = 100\,\mathrm{g}$, and the pigeon blacks out. The stone, which was at the highest point of its trajectory at that moment, flew with velocity $v = 30\,\mathrm{km\cdot h^{-1}}$ and lost half of its kinetic energy. Determine how far the pigeon's lifeless body falls from the spot where the rock was thrown. *Coo.*

First, we determine the horizontal distance from the point from which the stone was thrown to the collision point. We know that at that moment, the stone was at the highest point of its trajectory, so the vertical component of its velocity was zero. From this, we can calculate the time from the moment the rock was thrown as

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}\,.$$

The horizontal component of the velocity will remain constant throughout this entire period, so the stone will travel the distance

$$d = v\sqrt{\frac{2h}{g}}\,.$$

in the horizontal direction. Now we focus on the collision. If the stone had lost half of its kinetic energy, it moved after the collision with a velocity v', which we determine as

$$\frac{1}{2}mv'^2 = E'_\mathrm{k} = \frac{1}{2}E_\mathrm{k} = \frac{1}{2}m\left(\frac{v}{\sqrt{2}}\right)^2 \quad \Rightarrow \quad v' = \frac{v}{\sqrt{2}}\,.$$

Therefore, from the law of conservation of momentum for the velocity of a lifeless body of the pigeon V, the following will hold

$$MV + mv = MV' + m\frac{v}{\sqrt{2}} \quad \Rightarrow \quad V' = V + \frac{mv}{M}\frac{2-\sqrt{2}}{2}\,.$$

At this velocity, the pigeon will fly for $\sqrt{2h/g}$, for a total horizontal distance

$$s = \left(V + \frac{mv}{M}\frac{2-\sqrt{2}}{2}\right)\sqrt{\frac{2h}{g}}\,.$$

So the answer to the question in the problem statement is

$$s + d = \left(V + \frac{mv}{M}\frac{2-\sqrt{2}}{2} + v\right)\sqrt{\frac{2h}{g}} = 36\,\mathrm{m}\,.$$

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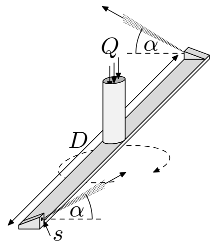
Problem EG ... dishwasher



Inside the dishwasher is a rotating propeller, into which water flows with the volumetric flow rate Q through its center. The water causes the propeller to rotate in a horizontal plane and flows radially inside it in tubes with cross-sectional area S on both sides. At the ends of the propeller are holes with cross-sectional area s, through which the water sprays and washes the dishes. The direction of the water is tangential to the rotation and forms an angle α with the horizontal plane. What is the angular frequency of the propeller if its diameter is D?

Jarda cleaned the dishes after party, and his head was still spinning.

Let us first consider the system associated with the propeller. Water flows into the propeller with a flow rate of Q, and as usual, all water that flows into the propeller along the axis of symmetry must also flow out through the holes at the ends. Let us denote the speed at which water flows out as v and the cross-sectional area of one hole as s. Thus, the flow rate from each hole at the end is

$$v_{\mathrm{out}} = \frac{Q}{2s}\,,$$

since Q is the flow rate for both halves of the propeller.

By changing the direction of the water from radial to tangential at the ends of the propeller, the water changes its angular momentum. And consequently changing the angular momentum of the propeller. The torque, i.e., the change in the angular momentum of the whole propeller, is

$$M_1 = \frac{\mathrm{d}L_1}{\mathrm{d}t} = 2\frac{RQ\cos\alpha}{2s}\frac{Q\rho}{2}\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{Q^2\rho R\cos\alpha}{2s}\,.$$

However, in our reference frame, the Coriolis force acts on the water flowing in the tube. It acts perpendicular to the direction of the water flow, thus creating an angular momentum. On the element of the water with a length $\mathrm{d}x$, which is at a distance x from the center, acts a torque with magnitude $\mathrm{d}M_2 = x2\omega\frac{Q}{2S}S\rho\,\mathrm{d}x$. Here $Q/2S$ denotes the radial velocity of the water in the propeller. The total torque due to the Coriolis force is

$$M_2 = 2\int_0^R x\omega Q\rho\,\mathrm{d}x = R^2\omega Q\rho\rho,.$$

Since we are interested in a steady state of the system and these two torques act against each other, by equating them: $M_1 = M_2$, we can obtain the rotation frequency as

$$\omega = \frac{Q\cos\alpha}{Ds}\,,$$

where we substituted for $R = D/2$.

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Problem EH ... watering the flower bed

Jarda's garden hose ends with a "gun" that divides the water beam into many smaller ones. Suppose that these small beams originate from a hemispherical cap at the end of this gun and that water gushes from all its points at the same velocity in a direction perpendicular to the tangent plane at that point. When Jarda lets the water squirt out of the gun pointing straight up, holding it at a height h, the water hits the surface A. How large of an area will the water fall on if he points the gun horizontally? The radius of the cap is small compared to the other sizes.

The garden is a highly inspiring environment for Jarda.

When Jarda holds the hose upwards, the water falls on the surface A, which is a circle of radius r. We can define this radius as the maximum possible distance that water can reach for each nozzle of a single hose with the same velocity at the same height h. Assume that this distance is reached for some inclination angle φ_0. The points with this angle form a circle on the hemispherical cap. When Jarda turns the hose such that it points horizontally, only half the number of points of this circle will be at the current position of the cap. At the same time, the water in that semi-plane will splash above and below the gun, but no longer backward. The resulting area on which the water falls will therefore be half of a circle with the same radius r, so the area will be $A/2$.

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Problem FA ... too fast electron

What atomic number would an element have to have for the ground state speed of an electron in a shell in order to exceed the speed of light, if we do not consider relativistic corrections? Use the Bohr model of the atom and assume that the atom is ionized to exactly one electron.

Jarda can't keep track of how much work there is in FYKOS.

In the Bohr model of the atom, we assume that the electron orbits the nucleus of the atom along a path that is curved by the electrostatic interaction between the nucleus and the electron. This results in a circular motion similar to the orbit of the planets around the Sun.

However, as observed, atoms can emit light only at certain wavelengths. The energy of this emitted light depends on the change in the radius of the trajectory. Since the spectrum of these radiated energies is discrete, electrons can orbit the nucleus only at certain discrete distances.

So much for the introduction. In the Bohr model of the atom, it is postulated that electrons orbit only on orbits on which the magnitude of the angular momentum is $L = n\hbar$, where n is the number of the shell on which the electron orbits, and $\hbar = h/2\pi$ is the reduced Planck constant. For the ground state of the electron we have $n = 1$.

In a force interaction, the magnitude of the centrifugal force is equal to the attractive electrostatic interaction

$$F_\mathrm{d} = m_\mathrm{e}\frac{v^2}{r} = F_\mathrm{e} = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2}\,,$$

where m_e is the mass of the orbiting electron, v is the orbital velocity, r is the distance from the nucleus, Z is the number of protons in the nucleus, ε_0 is vacuum permitivitty, and e is the elementary charge.

Now, we use the aforementioned condition for the magnitude of the angular momentum, which is

$$L = m_\mathrm{e} r v = n\hbar\,.$$

From this equation, we express r and insert it into the previous equation. This gives us

$$v = \frac{Ze^2}{4\pi\varepsilon_0 n\hbar}\,.$$

After substituting for $v = c$, $n = 1$ and expressing Z. We obtain

$$Z = \frac{4\pi\varepsilon_0\hbar c}{e^2} \doteq 137.065\,.$$

The inverse of this result is called the fine-structure constant. Now we need to proceed carefully – the charge of the nucleus has to be an integer. But if we substitute $Z = 137$ into the formula for the speed, we get the orbital velocity $v = 2.997\cdot 10^8\,\mathrm{m\cdot s^{-1}}$ which is, however, a lower value than $c = 2.998\cdot 10^8\,\mathrm{m\cdot s^{-1}}$. Therefore, the charge we are looking for has to be even higher, hence the answer is $Z = 138$.

An element with so many protons has not been observed yet. Moreover, according to the assumptions made in the problem statement, it would have to be 137 times ionized, which is a lot. Furthermore notice that the result is not far away from the atomic numbers of the heavy elements, for which special relativity has to be accounted for.

Finally, let us note that the Bohr model of the atom gives, in first approximation, the same structure of the electron shell as more advanced quantum models, but it does not describe the physical nature correctly.

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Problem FB ... annihilation peak

Far from a material with a thickness t, we will place a source of gamma rays with an energy of approximately 2 MeV. The radiation has an absorption coefficient of mu_1 in the material. For radiation of this energy can happen that a photon converts into an electron-positron pair inside the material (the loss in intensity due to this effect is already included in the μ_1 coefficient). However, the positron immediately annihilates with an electron and produces 2 gamma flashes, each with energy $E_\gamma = 511\,\mathrm{keV}$. The absorption coefficient of the material for the radiation with this energy is μ_2. We place a detector far behind the material and observe the intensity of radiation of the energy E_γ. For what thickness of material t will this intensity be the largest?
Hint: The infinitesimal loss of gamma-ray intensity in a material is proportional to the magnitude of the intensity, the absorption coefficient, and the unit of distance.

Jarda keeps losing photons somewhere.

From the hint, we derive the relation according to which the intensity I (i.e., the number of photons in the radiation beam) decreases as a function of the distance in the material. Namely, $-\,\mathrm{d}I = \mu_1 I\,\mathrm{d}x$, which is a differential equation that we solve by separation of variables to get the expected relation

$$I(x) = I_0 \mathrm{e}^{-\mu_1 x}\,,$$

where I_0 is the intensity before entering the material and x is the depth at which the intensity is calculated.

Some of this radiation is converted into an electron-positron pair. We assume that the probability of this effect does not depend on the position of the material, so the intensity of the E radiation is proportional only to $I(x)$. Thus, most of this radiation is produced at the beginning of the material, and the least at the end.

Once a gamma ray of energy E is produced and begins propagating through the material, it is also attenuated, now with a coefficient of μ_2. Let the intensity of this radiation $\mathrm{d}J_0$ occur in the $\mathrm{d}x$-neighbourhood of the x coordinate. According to the formula above, this intensity drops after exiting the material to

$$\mathrm{d}J = \mathrm{d}J_0 \mathrm{e}^{-\mu_2 (t-x)} ,$$

since the radiation has now passed a distance $t - x$ in the material.

But the resulting intensity $\mathrm{d}J_0$ is proportional to $I(x)$ and the length $\mathrm{d}x$ through some proportionality coefficient α, which we write as

$$\mathrm{d}J_0 = \alpha I(x)\,\mathrm{d}x = \alpha I_0 \mathrm{e}^{-\mu_1 x}\,\mathrm{d}x .$$

Substituting into the previous equation and integrating over the entire thickness of the material gives the total radiation intensity with energy E as

$$J = \alpha I_0 \int_0^t \mathrm{e}^{-\mu_1 x} \mathrm{e}^{-\mu_2 (t-x)}\,\mathrm{d}x = \alpha I_0 \mathrm{e}^{-\mu_2 t} \int_0^t \mathrm{e}^{(\mu_2 - \mu_1) x}\,\mathrm{d}x = \frac{\alpha I_0 \mathrm{e}^{-\mu_2 t}}{\mu_2 - \mu_1} \left(\mathrm{e}^{(\mu_2 - \mu_1) t} - 1 \right) .$$

The maximum intensity is found by a derivative with respect to t

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\alpha I_0}{\mu_2 - \mu_1} \left(-\mu_2 \mathrm{e}^{-\mu_2 t} \left(\mathrm{e}^{(\mu_2 - \mu_1) t} - 1 \right) + \mathrm{e}^{-\mu_2 t} \left(\mu_2 - \mu_1 \right) \left(\mathrm{e}^{(\mu_2 - \mu_1) t} \right) \right) ,$$

which we set equal to zero. We get

$$0 = -\mu_2 \mathrm{e}^{(\mu_2 - \mu_1) t_\mathrm{m}} + \mu_2 + \left(\mu_2 - \mu_1 \right) \mathrm{e}^{(\mu_2 - \mu_1) t_\mathrm{m}} ,$$

from which

$$\mu_1 \mathrm{e}^{(\mu_2 - \mu_1) t_\mathrm{m}} = \mu_2 ,$$

and now we can simply express the thickness for which the intensity is extremal as

$$t_\mathrm{m} = \frac{1}{\mu_2 - \mu_1} \ln \left(\frac{\mu_2}{\mu_1} \right) .$$

Looking at the relation $J(t)$, it can be seen that J is positive for all (positive) t, and that for points $t = 0$ and $t = \infty$, J decreases to zero. Thus, in computing the zero derivative of $J(t)$, we have indeed found the maximum of this function.

Regardless of which of the absorption coefficients is larger, it is clear that the thickness t_m is positive. If μ_2 is close to μ_1, our relation also holds in this limit.

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Problem FC ... evil chandelier

When Jarda is at home, he often happens to bump his head against the chandelier in the kitchen. The chandelier has the shape of a cylindrical shell without bases, a height of $h = 27\,\mathrm{cm}$, and a radius of $R = 12\,\mathrm{cm}$. In the upper part of the chandelier, a massless bar runs across its diameter. At the center of this bar, a rope by which the chandelier hangs from the ceiling is attached. The length of the rope is $l = 42\,\mathrm{cm}$. What is the period of the small oscillations of the chandelier when Jarda deflects it?

To find the period of the small oscillations we use the relation for a physical pendulum

$$T = 2\pi\sqrt{\frac{J}{mgd}}\,,$$

where J is the moment of inertia of the oscillating body with respect to the axis of rotation, m is the mass of the body, g is the gravitational acceleration and d is the distance of the center of gravity from the axis of rotation. The distance of the center of gravity from the axis of rotation is clearly $d = l + h/2$.

The moment of inertia with respect to the axis of rotation is given by Steiner's theorem as

$$J = J_T + md^2\,,$$

where J_T is the moment of inertia with respect to the axis running through the center of gravity of the body and has to be determined by integration.

Consider a thin circular element of the chandelier of thickness $\mathrm{d}x$ at a distance of x from the center of the chandelier and mass $\mathrm{d}m = m\,\mathrm{d}x/h$, where m is the mass of the whole chandelier. Relative to the center and the axis passing perpendicularly to the axis of symmetry, this element has a moment of inertia

$$\mathrm{d}J_T = \frac{1}{2}R^2\,\mathrm{d}m + x^2\,\mathrm{d}m\,,$$

where we used again Steiner's theorem. The factor 1/2 appears due to the moment of inertia of the thin ring with respect to the axis perpendicular to the axis of symmetry.

By integration we obtain

$$J_T = \frac{m}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{1}{2}R^2 + x^2\right)\mathrm{d}x = \frac{m}{h}\left[\frac{1}{2}R^2x + \frac{1}{3}x^3\right]_{-\frac{h}{2}}^{\frac{h}{2}} = m\left(\frac{1}{2}R^2 + \frac{1}{12}h^2\right)\,.$$

Thus, the solution to the problem is

$$T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + \frac{1}{12}h^2 + \left(l + \frac{h}{2}\right)^2}{g\left(l + \frac{h}{2}\right)}} = 1.53\,\mathrm{s}\,.$$

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Problem FD ... sphere in the road curve

Let us have a horizontal circular surface with the radius R. An edge of height h is raised around this circle. A sphere of mass m and radius r rolls along the edge while touching it. The friction between the sphere and all other surfaces is large enough not to make the sphere slip. Determine the kinetic energy of the sphere if one revolution in this enclosure takes time T. It holds that $r < h$ and $R > r$. *You do not need a steering wheel to turn.*

We have described the kinematics of the sphere. Let us denote the angular velocity of the center of gravity's movement as $\omega = 2\pi/T$. The center of gravity moves on a circle with a radius $R-r$; thus, its velocity is $v = \omega\,(R-r)$.

Now let us switch to a system that moves with velocity v in the direction of motion of the sphere. Let us draw the axis z perpendicular to the bottom plane, the axis x from the circle's center towards the sphere.

Consider rotation along the bottom plane. The sphere is stationary, and the bottom plane moves with velocity v. Since the sphere is moving without slipping, the velocity of the points on edge must be also v. Let us denote the angular velocity of rotation along the x axis as ω_x, then the following is true

$$\omega_x r = v = \omega\,(R-r)\ .$$

Alternatively, let us just consider rotation relative to the edge. The sphere is stationary, and the edge moves with a velocity ωR. Using the same argument, we get that the angular velocity of rotation with respect to the z axis is

$$\omega_z r = \omega R\,.$$

Therefore, the magnitude of the angular velocity vector will be $\sqrt{\omega_x^2 + \omega_z^2}$. Since the sphere has the same moment of inertia with respect to all its axes, we can express the rotational kinetic energy as

$$E_{\mathrm{k,r}} = \frac{1}{2} J\left(\omega_x^2 + \omega_z^2\right) = \frac{1}{2}\frac{2}{5} m r^2 \omega^2 \left(\left(\frac{R-r}{r}\right)^2 + \left(\frac{R}{r}\right)^2\right) = \frac{1}{2}\frac{2}{5} m\omega^2\left(2R^2 - 2Rr + r^2\right)\ .$$

This solves the rotation of the sphere, and we only need to add the kinetic energy of the translation to it, from which we get

$$E_{\mathrm{k}} = \frac{1}{2} m v^2 + E_{\mathrm{k,r}} = \frac{2\pi^2 m}{5T^2}\left(9R^2 - 14Rr + 7r^2\right)\ .$$

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Problem FE ... we will prevent armageddon

In August 2022, NASA conducted a test to see if a probe impacting an asteroid could affect its orbit. The chosen object was Dimorphos, a small asteroid orbiting its larger colleague called Didymos. The probe with a mass of $570\,\mathrm{kg}$ *shortened Dimorphos' orbital period around Didymos by 32 minutes. What's the maximum change in Didymos' orbital period if this probe crashed directly into Didymos? This asteroid orbits the Sun with a semi-major axis* $1.64\,\mathrm{au}$ *and* *eccentricity* 0.384*, has a mass of* $5.2\cdot 10^{11}$ kg*, and the probe would hit the surface with a velocity of* 22 000 km·h^{-1}*. The mass of Dimorphos (which was, by the way, discovered at Ondřejov Observatory here in the Czech Republic) is neglected in this problem. Assume no material was ejected from Didymos due to the collision.* *Jarda wants to move something heavy.*

We first state Kepler's third law, from which we get the period of Didymos T as

$$T = T_{\oplus}\sqrt{\frac{a^3}{a_{\oplus}^3}}\,,$$

where $T_{\oplus} = 1\,\mathrm{year}$ is the period of the Earth, a the lengh of Didymos' semi-major axis and $a_{\oplus} = 1\,\mathrm{au}$ the semi-major axis of the Earth's orbit. Thus we see that the period is proportional to the semi-major axis. Since the probe is much lighter than the asteroid, we can assume that the impact will only slightly change the asteroid's orbit. We, therefore, differentiate this relation to obtain the dependence of the change in the period on the change in the semi-major axis of Didymos' orbit

$$\Delta T = \frac{3}{2}T_{\oplus}\sqrt{\frac{a^3}{a_{\oplus}^3}}\frac{\Delta a}{a}\,.$$

As we know, the conservation of energy law applies for motion in a gravitational field. The total energy of an object (neglecting its mass relative to the central body) per unit of its mass is $E = -GM_{\odot}/(2a)$, where G is the gravitational constant, $M_{\odot}$ is in our case the mass of the Sun and a is the semi-major axis. Thus we see there is a simple relation between energy per unit of mass and the semi-major axis. We can also calculate this energy as the sum of a potential and a kinetic energy per unit of mass

$$E = \frac{1}{2}v^2 - \frac{M_{\odot}G}{r}\,.$$

By comparing these two parts we get

$$-\frac{GM_{\odot}}{2a} = \frac{1}{2}v^2 - \frac{M_{\odot}G}{r}\,.$$

This equation has to hold before, as well as after the collision but with different a and v, because r is not changed by the collision. Therefore, we can express the change in the semi-major axis using the change in the asteroid's speed by differentiating this equation

$$\frac{GM_{\odot}}{2a^2}\Delta a = \mathbf{v}\cdot\Delta\mathbf{v}\,.$$

To maximize the change in the semi-major axis, we require $\mathbf{v}$ and $\Delta\mathbf{v}$ to point in the same or the opposite direction.

Let us now consider for a moment the change in Didymos' velocity $\Delta\mathbf{v}$ due to the collision. We can find that using the law of conservation of momentum. This change in velocity will be the same as in the reference frame of the Sun and the whole planetary system, as well as in the frame of reference where Didymos is originally at rest. There the following holds

$$m\mathbf{u} = (m+M)\,\Delta\mathbf{v}\,.$$

Now we can insert into the previous relation

$$\frac{GM_{\odot}}{2a^2}\Delta a = v\frac{mu}{M+m}\,.$$

At the right-hand side of the equation, we can neglect m in the denominator relative to M. The change of the semi-major axis is therefore

$$\Delta a = \frac{2muva^2}{GM_\odot M}\,.$$

Now we want to maximize v so that Δa is as large as possible. The asteroid has a maximum magnitude of velocity when it is closest to the Sun, which occurs at a distance $a\,(1-e)$. There, the magnitude of its velocity is

$$v = \sqrt{\frac{M_\odot G}{a}\left(\frac{2}{1-e} - 1\right)} \doteq 35\,\mathrm{km{\cdot}s^{-1}}\,.$$

By substituting for all unknown parameters we get

$$\Delta T = \frac{3}{2} T_\oplus \sqrt{\frac{a^3}{a_\oplus^3}}\frac{2mavu}{GM_\odot M} = 0.086\,\mathrm{s}\,.$$

The orbital period of Didymos around the Sun would change by 0.086 s. For simplicity, we did not consider the effect of material ejection due to the probe hitting the surface of the body in the problem statement. However, the fragments produced by the impact would also have some momentum, which on average would be in the opposite direction to the probe. Thus, the change in momentum (and, therefore, as a consequence, the orbital period) would be higher.

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Problem FF ... two rings

Two rings made out of a thin conductive wire lie on a common axis at a distance of $z = 15\,\mathrm{cm}$ from each other. One ring has a radius of $a = 5\,\mathrm{cm}$, while the other has a radius of $b = 2\,\mathrm{mm}$, you can therefore consider that $b \ll a$. Of course, both rings can act as coils. Determine the mutual inductance of the rings. *Jindra wears two rings on one finger.*

The mutual inductance is the same whether the first ring acts on the second one or vice versa. The magnetic induction on the axis of the bigger ring at a distance of z is

$$B = \frac{\mu_0 I_1 a^2}{2(z^2 + a^2)^{3/2}}\,,$$

where I_1 is the current in the bigger ring. Since the second ring has a significantly smaller radius than the first ring, the magnetic flux through the second ring can be approximated as

$$\Phi_{12} = B\pi b^2 = \frac{\pi\mu_0 I_1 a^2 b^2}{2(z^2 + a^2)^{3/2}}\,.$$

The mutual induction of the two coils is then

$$M = \frac{\Phi_{12}}{I_1} = \frac{\pi\mu_0 a^2 b^2}{2(z^2 + a^2)^{3/2}} = 5\cdot 10^{-12}\,\mathrm{H}\,.$$

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Problem FG ... stone doesn't fall far from pigeon

A pigeon with a head at height $h = 20\,\mathrm{cm}$ is eating bread crumbs from the ground. Suddenly, a rock hits the ground at a distance r from its head, and the pigeon flies away with a velocity $v = k/r^2$ away from the rock (the direction is taken from the point of the impact to the pigeon's head), where k is a constant. Determine the value of r for the pigeon to fly as far as possible from its initial position if it exerts no additional force to stay in the air as it moves.

Vojta really doesn't mind pigeons.

The pigeon's flight is nothing but an oblique throw. A distance reached by a mass point thrown at an angle α is given by

$$d = \frac{v^2}{g} 2 \sin\alpha \cos\alpha$$

for initial velocity $v = k/r^2$. The sine of the angle α is determined from simple goniometry as h/r and the cosine as $\sqrt{r^2 - h^2}/r$. Plugging everything into the equation, we obtain

$$d = 2\frac{k^2 h}{g} \frac{\sqrt{r^2 - h^2}}{r^6} = 2\frac{k^2 h}{g} \sqrt{\frac{r^2 - h^2}{r^{12}}}\,.$$

So, we just need to find the maximum (which, by physics intuition, should exist) of the expression

$$\mathcal{D} = \frac{r^2 - h^2}{r^{12}}\,.$$

We set the derivative of the expression equal to zero, resulting in

$$\mathcal{D}' = \frac{12h^2 - 10r^2}{r^{13}} = 0 \quad \Rightarrow \quad r_{\max} = \frac{\sqrt{30}}{5} h\,,$$

which gives us that the pigeon flies the farthest when the rock hits at a distance $r_{\max} \doteq 21.9\,\mathrm{cm}$.

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Problem FH ... love and truth will overcome lies and hatred

Freedom of speech has a speed $v = 1.00\,\mathrm{m{\cdot}s^{-1}}$ and flies through space toward a better society. However, it hits hard against the immovable wall the local regime has erected against freedom and elastically reflects. In the opposite direction, however, it travels only the distance $l_1 = 1.0\,\mathrm{m}$ before it hits a second wall, which is approaching the first wall at speed $u = 0.01\,\mathrm{m{\cdot}s^{-1}}$. The regime began to repress freedom of speech. Free speech is elastically reflecting from the walls and trying to break free from this clench. Since the world is supposed to end well, it will break its prison if it starts to apply an average force $F = 10\,\mathrm{N}$ to the moving wall. The question, however, is how long it will take to break out after the first hit. Think of the force F as the average change in momentum for two successive impacts over the time elapsed between them. The mass of freedom is $m = 1.0\,\mathrm{g}$.

Jarda feels under pressure.

Let us denote by Δp_n the change in momentum of the free speech at the nth impact to the moving wall, and t_n the time that elapses between nth and $n+1$st impact into the moving wall. Then according to the problem statement, the average force is

$$F_n = \frac{\Delta p_n + \Delta p_{n+1}}{2t_n}\,.$$

Now we have to calculate p_n and t_n.

We solve the whole problem in a system where the speed of freedom was v. After the first impact into the immutable wall, the freedom of speech moves with the same speed in the opposite direction. After each reflection from the moving wall, the speed increases by $2u$. We easily justify that by moving to the system where this wall is at rest. Here, the speed of freedom of speech is $v+u$, after the elastic reflection is the speed just reversed. This system, however, was moving at a speed u compared to the original one, so after the reflection is the speed of freedom of speech in the original system $v+2u$. This is true for every reflection on this wall, therefore the speed after the nth reflection from the moving wall is $v+2nu$.

We calculate the time elapsed between two reflections on this moving wall. We denote the distance between walls after the nth reflection by l_n (we know the distance l_1 after the first impact from the problem statement). For the $n+1$st reflection the following condition must hold

$$(v+2nu)\,t_n + u\;t_n = 2l_n \quad\Rightarrow\quad t_n = \frac{2l_n}{v+(2n+1)\,u}\,,$$

which determines when the next reflection occurs.

We see that t_n depends on l_n, so we have to find it. Between the nth and the following reflection the distance between the two walls decreases by ut_n, so

$$l_{n+1} = l_n - u\;t_n = l_n\frac{v+(2n-1)\,u}{v+(2n+1)\,u}\,.$$

Let us write this expression also for l_n as

$$l_n = l_{n-1}\frac{v+(2n-3)\,u}{v+(2n-1)\,u}$$

and let us insert it into the equation above. We get

$$l_{n+1} = l_{n-1}\frac{v+(2n-3)\,u}{v+(2n-1)\,u}\frac{v+(2n-1)\,u}{v+(2n+1)\,u} = l_{n-1}\frac{v+(2n-3)\,u}{v+(2n+1)\,u}\,.$$

Several elements have been eliminated. We can iterate this procedure and we get

$$l_n = l_1\frac{v+u}{v+(2n-1)\,u}\,.$$

Now we know how t_n depends on n. Let us now express Δp_n as

$$\Delta p_n = m\left(v+2nu-\left(-v-2\left(n-1\right)u\right)\right) = m\left(2v+2u\left(2n-1\right)\right)\,.$$

Then we get the final relation for the force F_n as

$$F_n = 2m\frac{v+2nu}{t_n} = m\frac{\left(v+(2n-1)\,u\right)\left(v+2nu\right)\left(v+(2n+1)\,u\right)}{l_1\left(u+v\right)}\,.$$

Now we check the values given to us. For small n we can neglect the elements with u compared to v and we get a force on the order of $10^{-3}\,\mathrm{N}$. Presumably then, the n needs to be very large for the freedom of speech to exert such a large average force. But then we can put $2n-1 \approx 2n+1 \approx 2n$ and express

$$F_n \doteq m\frac{\left(v+2nu\right)^3}{l_1\left(u+v\right)} \quad\Rightarrow\quad n \doteq \frac{\sqrt[3]{\frac{Fl_1(u+v)}{m}}-v}{2u} = 1\ 031\,.$$

If we insert $n = 1\ 030$, we get $F_{1\ 030} \doteq 9.98\,\mathrm{N}$, for $n = 1\ 031$ it is already $F_{1\ 031} \doteq 10.01\,\mathrm{N}$. Therefore, there must be $n = 1\ 031$ reflections on the moving wall.

The time we are looking for is then obtained from the knowledge of l_n and the motion of the wall as

$$t = \frac{l_1 - l_{1031}}{u} + \frac{l_1}{v} = \frac{l_1}{u}\frac{2\,(1\ 031 - 1)\,u}{v + (2\cdot 1\ 031 - 1)\,u} + \frac{l_1}{v} \doteq 96\,\mathrm{s}\,.$$

Note also that for $nu \gg v$ is $ml'v' = ml_1v$, where $v' = 2nu$ and l' is the distance of the walls at the speed of freedom of speech v'. Thus, the law of conservation of some kind of angular momentum holds.

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Problem GA ... running princess

The princess was walking around the castle when she suddenly noticed a spider crawling up her veil at $u = 1.00\,\mathrm{m\cdot s^{-1}}$ towards her. She screamed terribly (which of course didn't help her at all) and started running away at $v = 3.00\,\mathrm{m\cdot s^{-1}}$ (which also didn't help her at all, as she was dragging the veil behind her...). She ran through the door like that, which slammed shut just behind the spider. At that point, $l_0 = 3.00\,\mathrm{m}$ of the veil remained between the spider and the princess.

Both the princess and the spider continued to move, and the slammed veil began to stretch perfectly. The princess thought she could escape the spider because she was running faster than it, but she was wrong. In what time t_f will the spider catch up with the princess? Think of the spider as a point. *Legolas likes spiders.*

With this problem, the hardest part is finding a way to approach it mathematically. In my opinion, the most efficient way is to introduce the quantity p, which we will use to denote the ratio of the section of the veil that the spider has already passed and the current length of the veil, where we denote the current length of the veil l and $l = l_0 + vt$ will hold, where t is the time since the door was slammed.

In the beginning, $p = 0$. The spider catches up with the princess when $p = 1$.

This approach is advantageous because p is not directly affected by the princess's running since the veil extends the same along its entire length, and thus if the spider had remained stationary, p would not have changed at all.

However, when the spider moves, it travels a distance $u\,\mathrm{d}t$ in a small instant $\mathrm{d}t$. So in this instant p increases by an element $\mathrm{d}p = u\,\mathrm{d}t/l$.

We get a differential equation

$$\mathrm{d}p = \frac{u\,\mathrm{d}t}{l_0 + vt}\,,$$

which is already in the form of separated variables, i.e., we only need to integrate and express t_f

$$\int_0^1 \mathrm{d}p = \frac{u}{v}\int_0^{t_\mathrm{f}} \frac{\mathrm{d}t}{l_0/v + t}\,,$$

$$[p]_0^1 = \frac{u}{v}\left[\ln\left(\frac{l_0}{v} + t\right)\right]_0^{t_\mathrm{f}} = \frac{u}{v}\left(\ln\left(\frac{l_0}{v} + t_\mathrm{f}\right) - \ln\left(\frac{l_0}{v}\right)\right)\,,$$

$$\frac{v}{u} = \ln\left(1 + \frac{t_\mathrm{f} v}{l_0}\right)\,,$$

$$\mathrm{e}^{\frac{v}{u}} = 1 + \frac{t_\mathrm{f} v}{l_0}\,,$$

$$\left(\mathrm{e}^{\frac{v}{u}} - 1\right)\frac{l_0}{v} = t_\mathrm{f}\,.$$

We can observe that for v approaching 0 in the limit, we can write $\mathrm{e}^{v/u} = 1 + v/u$, and then we get $t_\mathrm{f} = l_0/u$. Substituting the values from the problem statement, we get $t_\mathrm{f} = 19.1\,\mathrm{s}$.

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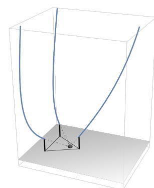
Problem GB ... stone is falling among pigeons



Three pigeons of height $h = 20\,\mathrm{cm}$ are standing at the vertices of an equilateral triangle with side length $a = 50\,\mathrm{cm}$ and eating bread crumbs. Suddenly, a stone falls among them on one of the triangle's medians such that the point of impact divides the median in the ratio $2:1$, and it is not the triangle's centroid. All the pigeons fly away with an initial velocity $v = k/r^2$ away from the stone (the direction is taken from the point of impact to the bird's head), where k is a constant and r is the initial distance of the bird's head from the point of impact. At this point, however, group behavior kicks in for the pigeons. At each moment, each bird instinctively averages the velocity vectors of its two fellow pigeons and chooses an acceleration such that after a period $T = 3\,\mathrm{s}$ of uniformly accelerated motion, it moves at that average velocity. After some time, the movement of the feathered friends becomes steady, and they all fly in the same direction at the same speed. Determine the angle that this direction forms with the ground. *Vojta wonders how a pigeon works.*

Let us denote the velocities of the pigeons $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ and their accelerations correspondingly. The heuristic that the pigeons follow can be mathematically rewritten as

$$\mathbf{v_1} + \mathbf{a_1}T = \frac{1}{2}\left(\mathbf{v_2} + \mathbf{v_3}\right)\,,$$

$$\mathbf{v_2} + \mathbf{a_2}T = \frac{1}{2}\left(\mathbf{v_1} + \mathbf{v_3}\right)\,,$$

$$\mathbf{v_3} + \mathbf{a_3}T = \frac{1}{2}\left(\mathbf{v_1} + \mathbf{v_2}\right)\,.$$

If we add these three equations, we get an interesting equation

$$\left(\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3}\right)T = 0\,.$$

This tells us that the total instantaneous acceleration of the system of three pigeons is zero. This means that the whole system is either at rest or moving in a uniform linear motion – that is, the sum of the velocities of the three birds is constant throughout the motion. This means that the total velocity of the flock of pigeons after the direction is steady is determined as the average of the velocity vectors at the beginning of the movement, i.e.,

$$\mathbf{V} = \frac{1}{3}\left(\mathbf{v_1}(0) + \mathbf{v_2}(0) + \mathbf{v_3}(0)\right) ,$$

Now we only need to solve the initial conditions. Let's introduce a Cartesian coordinate system with the origin at the center of the triangle, the z-axis perpendicular to the ground, and the first pigeon standing on the x-axis. The positions of the heads of the pigeons $\mathbf{h_i}$ and the position of the stone $\mathbf{s}$ can then be determined from simple geometry as

$$\begin{aligned}
\mathbf{h_1} &= \left[\frac{\sqrt{3}}{3}a, 0, h\right] ,\\
\mathbf{h_2} &= \left[-\frac{\sqrt{3}}{6}a, \frac{1}{2}a, h\right] ,\\
\mathbf{h_3} &= \left[-\frac{\sqrt{3}}{6}a, -\frac{1}{2}a, h\right] ,\\
\mathbf{s} &= \left[\frac{\sqrt{3}}{6}a, 0, 0\right] ,
\end{aligned}$$

from which we can easily determine the directions and magnitudes of the velocities $\mathbf{v_i}$ at time zero as

$$\begin{aligned}
\mathbf{v_1}(0) &= \frac{k}{||\mathbf{h_1} - \mathbf{s}||^3}\left(\mathbf{h_1} - \mathbf{s}\right) \doteq k \cdot \begin{pmatrix} 9.620 \\ 0.000 \\ 13.33 \end{pmatrix} \mathrm{m}^{-2} ,\\
\mathbf{v_2}(0) &= \frac{k}{||\mathbf{h_2} - \mathbf{s}||^3}\left(\mathbf{h_2} - \mathbf{s}\right) \doteq k \cdot \begin{pmatrix} -3.604 \\ 3.121 \\ 2.497 \end{pmatrix} \mathrm{m}^{-2} ,\\
\mathbf{v_3}(0) &= \frac{k}{||\mathbf{h_3} - \mathbf{s}||^3}\left(\mathbf{h_3} - \mathbf{s}\right) \doteq k \cdot \begin{pmatrix} -3.604 \\ -3.121 \\ 2.497 \end{pmatrix} \mathrm{m}^{-2} .
\end{aligned}$$

The sought vector is, therefore, equal to

$$\mathbf{v} = k' \begin{pmatrix} 2.413 \\ 0.000 \\ 18.32 \end{pmatrix} ,$$

where k' is a certain constant. Now we need to determine the angle that this vector makes with the ground, which we do simply by using goniometry because

$$\tan\varphi = \frac{v_z}{v_x} \doteq \frac{18.32}{2.413} \doteq 7.59 .$$

From this, we determine the angle $\varphi \doteq 82.5^\circ$.

Note that the entire situation can also be resolved in an exact way, which implies, among other things, the validity of the assumption that the speed of the feathered friends will stabilize. One of the methods to resolve this situation is given below.

The original triple of equations of motion can also be rewritten in matrix form, but we must remember that all the elements are vectors themselves.

$$\begin{pmatrix}\mathbf{a_1}\\ \mathbf{a_2}\\ \mathbf{a_3}\end{pmatrix} = \begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix}' = \frac{1}{2T}\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix}$$

At this point, our goal is to solve a system of linear differential equations. For this purpose, we need to diagonalize the matrix shown above. This matrix has eigenvalues 0, -3 and -3, which are associated with eigenvectors

$$M_0 = \operatorname{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \qquad M_{-3} = \operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}.$$

So we can rewrite our set in the form

$$\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix}' = \frac{1}{2T}\begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}\begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix}.$$

Now, if we introduce substitution

$$\begin{pmatrix}\mathbf{u_1}\\ \mathbf{u_2}\\ \mathbf{u_3}\end{pmatrix} = \begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix},$$

we can further simplify our set of equations to

$$\begin{pmatrix}\mathbf{u_1}\\ \mathbf{u_2}\\ \mathbf{u_3}\end{pmatrix}' = \frac{1}{2T}\begin{pmatrix}0 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}\begin{pmatrix}\mathbf{u_1}\\ \mathbf{u_2}\\ \mathbf{u_3}\end{pmatrix},$$

which are already three simple separable differential equations that we can solve. We can directly write

$$\begin{pmatrix}\mathbf{u_1}\\ \mathbf{u_2}\\ \mathbf{u_3}\end{pmatrix} = \begin{pmatrix}\mathbf{c_1}\\ \mathbf{c_2}\mathrm{e}^{-\frac{3}{2T}t}\\ \mathbf{c_3}\mathrm{e}^{-\frac{-3}{2T}t}\end{pmatrix},$$

from where, by reverse substitution

$$\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix} = \begin{pmatrix}\mathbf{c_1} - (\mathbf{c_2} + \mathbf{c_3})\,\mathrm{e}^{-\frac{3}{2T}t}\\ \mathbf{c_1} + \mathbf{c_2}\mathrm{e}^{-\frac{3}{2T}t}\\ \mathbf{c_1} + \mathbf{c_3}\mathrm{e}^{-\frac{3}{2T}t}\end{pmatrix}, \tag{1}$$

where the vectors $\mathbf{c_1}$, $\mathbf{c_2}$, and $\mathbf{c_3}$ are vector integration constants that we determine from the initial conditions. Before we find these initial conditions, note that after a sufficiently long time, all components that do not depend on $\mathbf{c_1}$ disappear, since

$$\lim_{t\to\infty}\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\end{pmatrix} = \begin{pmatrix}\mathbf{c_1}\\ \mathbf{c_1}\\ \mathbf{c_1}\end{pmatrix} \,,$$

which indeed corresponds to the assumption that pigeons "converge" to the same velocity, and we can consider our intuition correct. So we only need to find the vector $\mathbf{c_1}$. At time $t = 0\,\mathrm{s}$, from equation (1), the following holds

$$\mathbf{c_1} = \frac{1}{3}\left(\mathbf{v_1}(0) + \mathbf{v_2}(0) + \mathbf{v_3}(0)\right) \,,$$

thus obtaining purely mathematically the same result as using the physical intuition above.

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Problem GC ... crashed charges

Two small particles, each of mass $m = 0.1\,\mathrm{g}$, are at rest in a vacuum at a distance $L = 1.0\,\mathrm{m}$ apart. One of them has charge $Q = 0.1\,\mathrm{\mu C}$, the other the same charge of the opposite sign. How long will it take for them to reach each other? You can neglect the loss of energy due to bremsstrahlung. *Robo wanted to annihilate everything, but all he got was a collision.*

First of all, we need to realize that the sizes of the particles are much smaller than their mutual distances; thus, we can say that they will reach each other when their distance equals 0. The two particles are the same, which means they will move with the same acceleration, which implies that each particle must travel a distance $L/2$. Using Kepler's third law, we can find the time it takes the particles to travel the distance $L/2$ to the system's center of gravity. The law describes that if two bodies orbit around the same point of mass on conic sections (ellipse, circle, ...) with semi-major and semi-minor axis a_1, a_2 and their orbital periods T_1, T_2, then the following is true

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3 \,.$$

The two particles will travel in straight lines to their mutual center of mass, which is $L/2$ far from each. The line segment is actually an ellipse with minor semi-axes equal to 0, with its focuses at the ends of the ellipse, and the major semi-axes have a length $L/4$. Half the orbital period along such an ellipse (segment) equals the time it takes a body to travel from one end of the segment to the other. From Kepler's third law, we know that its orbital period must be equal to the orbit of particles moving along a circle with a radius $L/4$ with the center at a mutual center of gravity. Therefore, we only need to calculate half of its orbital period along this circle. From the equilibrium of the electric and centrifugal forces, we will get

$$\frac{mv_0^2}{L/4} = \frac{Q^2}{4\pi\varepsilon_0(2\cdot L/4)^2} \,.$$

For the orbital period along the circle, the following holds

$$T = \frac{2\pi L/4}{v_0}\,,$$

from which we get

$$T^2 = \left(\frac{2\pi L/4}{v_0}\right)^2 = \frac{\varepsilon_0 m\,(\pi L)^3}{Q^2}\,.$$

Halving it gives us

$$t = \frac{T}{2} = \frac{\sqrt{\varepsilon_0 m\,(\pi L)^3}}{2Q}\,.$$

What numerically equals $t = 0.83\,\mathrm{s}$.

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