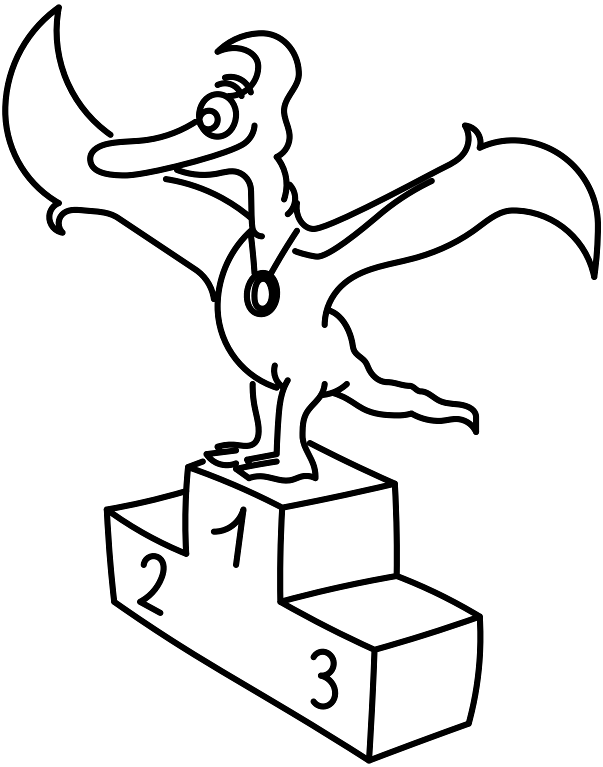
Solutions of problems

Fyziklani



Problem AA ... it flows

A reactor of power 250 MW *is cooled by a water pump with a volumetric flow rate of* 1 200 hℓ *per minute. If the water flows through the reactor only once, how much will its temperature rise? Do not consider heat loss.* *Pepa stole this from a textbook.*

The reactor power tells us that it releases heat $Q = 250\,\mathrm{MJ}$ per 1 second to its surroundings. During this second, $\Delta V = 20\,\mathrm{h\ell}$ of water of mass $m = \rho \Delta V$ flows through the reactor. We substitute that into the known formula $Q = mc\Delta T$ and express the change in the temperature of the flowed water as

$$\Delta T = \frac{Q}{c\rho\Delta V}\,,$$

which is approximately 30 K after number substitution.

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Problem AB ... muddy

The FYKOS-bird went sledging to a hillside. The hillside's perpendicular height is 10 m *and the path down the hill is* 20 m *long. However, there is already some grass and mud in the path which slows the sledge down, so the FYKOS-bird stops at the distance* 30 m *from the bottom of the hill after going downhill. What is the value of the coefficient of friction between the sledge and the ground? Neglect all other resistive forces.*
When Verča was young it was possible to go sledging in winter.

The friction force acting upon the sledge during the ride must do a work equal to the potential energy of the FYKOS-bird at the top of the hillside. During the ride from the hillside, the friction force $F_1 = mgf\cos\alpha$ acts upon the sledge, where f is the friction coefficient and α is the slope of the hill. We obtain the cosine of this angle from the Pythagorean theorem

$$\cos\alpha = \frac{\sqrt{s^2 - h^2}}{s}\,,$$

where h is the hillside's perpendicular height and s is the length of the path. Down below the hill the path is no longer inclined by any angle so the friction force is $F_2 = mgf$. If we denote the distance after which the FYKOS-bird stops as d, we get

$$W = E_\mathrm{p}\,,$$
$$F_1 s + F_2 d = mgh\,,$$
$$mgfs\cos\alpha + mgfd = mgh\,,$$
$$f = \frac{h}{s\cos\alpha + d} = \frac{h}{\sqrt{s^2 - h^2} + d}\,.$$

The final coefficient equals approximately $f \doteq 0.21$. It is independent of the weight of the FYKOS-bird and the gravitational acceleration.

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Problem AC ... headphone jack

By how many minutes can a smartphone's battery life be extended by omitting a headphone jack and having a larger battery? Headphone jack needs a volume equivalent to a 1.5 cm *long cylinder with a diameter of* 3.5 mm*. A typical battery has a volume of* 20 cm^3 *and provides* 15 h *of active usage.* *Matěj follows new smartphone trends.*

We assume that a battery capacity is directly proportional to its size. Therefore, using the given values, we obtain battery life per volume as

$$\frac{15\,\mathrm{h}}{20\,\mathrm{cm}^3} = 0.75\,\mathrm{h{\cdot}cm^{-3}}\,.$$

We determine the volume saved by omitting the jack as a volume of cylinder, thus

$$V = \pi \cdot \left(\frac{0.35}{2}\,\mathrm{cm}\right)^2 \cdot 1.5\,\mathrm{cm} \doteq 0.144\,\mathrm{cm}^3\,.$$

To find the upgrade of battery life, we multiply both values we obtained earlier

$$\Delta t \doteq 0.75\,\mathrm{h{\cdot}cm^{-3}} \cdot 0.144\,\mathrm{cm}^3 \doteq 0.11\,\mathrm{h} \doteq 6.5\,\mathrm{min}\,.$$

If a phone has a headphones jack, it loses only several minutes of battery life; therefore, it is not a relevant argument why manufacturers omit it. The more important reasons seem to be improved water resistance of the smartphone or increased profit from selling more wireless headphones.

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Problem AD ... Fyziklani Prize

The FYKOS organizers were thinking about the prize for the best teams at Fyziklani. One of the ideas was to issue commemorative coins. It would be 18-karat golden (the rest is silver) coins for the winners, whose value would be $c_1 = 31\,000$ *Kč apiece. The runner-up team would get silver coins, whose value would be* $c_s = 1\,100$ *Kč apiece. The silver coin is twice as heavy as the golden one. How many times is the unit price of gold greater than the unit price of silver? Assume zero minting costs. One karat is* 1/24 *of the total mass.*

Jarda won a keyboard with lighting effects once.

Let us denote the mass of the golden coin by m. The mass of the silver coins is then $2m$. From the definition of the karat, three quarters $(\frac{18}{24})$ of the mass of the golden coins is gold, i.e., $m_z = \frac{3}{4}m$.

Price of silver per unit mass is $j_s = \frac{c_2}{2m}$. Price of the gold per unit mass is

$$j_z = \frac{c_z}{m_z} = \frac{c_1 - j_s \frac{m}{4}}{\frac{3m}{4}}\,.$$

Hence, the ratio of the unit prices is

$$\frac{j_z}{j_s} = \frac{c_1 - \frac{c_2}{8}}{\frac{3c_2}{8}} = \frac{8c_1 - c_2}{3c_2} \doteq 74.8\,.$$

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Problem AE ... dang reduction

On highway with ongoing road work, the normal three lanes with maximum permitted speed $130\,\mathrm{km\cdot h^{-1}}$ are usually reduced to only two lanes with maximum speed restricted to $80\,\mathrm{km\cdot h^{-1}}$. How much (in percents) does the maximum number of cars that can drive on the highway in hour drop if we assume the traffic is optimal? Cars drive at the speed limit in all lanes and have time spacing $2\,\mathrm{s}$ between their front bumpers. Compare the highway on an ordinary day (no reduction) to a day with restrictions (with reduction).

Karel drove on the D1 (in the Czech Republic and Slovakia).

The solution is a bit surprising because if we have constant time spacing, one car passes one lane in time $\Delta t = 2\,\mathrm{s}$. Hence, if the speed of cars changes, distances between them change, but the total number of cars that pass the highway remains the same. If the allowed time spacing (which we measure between the front bumpers not to make the situation more complicated by the length of cars) is fully utilized, we get that the number of cars depends only on number of lanes. The proportion of cars that pass the highway under road work is therefore $2/3 \doteq 0.667 = 66.7\,\%$. Because we were asked by how much the capacity drops, the answer is $1 - 2/3 = 1/3 \doteq 0.333 = 33.3\,\%$.

If we assumed distance spacing between cars, the solution would be more complicated. However, it is recommended to maintain time spacing mainly due to the driver's reaction time, which remains constant regardless of the car's speed. Indeed, keeping the same space distancing could be seriously dangerous while driving at high speeds while it would be unduly cautious in the cities at lower speeds.

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Problem AF ... oh, those Newton's laws

What is the magnitude of drag force acting on the car, which is going on a straight road at constant velocity $30\,\mathrm{m\cdot s^{-1}}$ if the car has an engine of input power $150\,000\,\mathrm{W}$ that provides car with thrust $3\,000\,\mathrm{N}$? The temperature of surrounding air is $37\,^\circ\mathrm{C}$.

Robert just came up with this problem!

Because the car is driving at a constant speed, all forces acting on it must be in equilibrium (according to Newton's first law of motion). Drag forces are compensated by the force produced by engine. We do not assume any other forces acting on the car. All the other numbers are useless. Thus, the answer is $3\,000\,\mathrm{N}$.

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Problem AG ... toilet vandalism

A toilet paper roll has a cylindrical shape with height $v = 9.4\,\mathrm{cm}$, and outer diameter $D_2 = 12.5\,\mathrm{cm}$. The hollow cardboard cylinder in its center is of diameter $D_1 = 4\,\mathrm{cm}$. Thickness of individual paper rectangles is $h = 0.5\,\mathrm{mm}$. What is the minimum number of toilet paper rolls to cover the entire Earth's surface by a single layer of toilet paper if we assume that the paper is wound up tightly and that we can approximate the spiral winding of paper on a cardboard *cylinder with cylindrical layers? Round the result to two significant digits.*

Robo thinks about problems in unusual places.

By using the provided approximation, we know that a toilet paper roll has

$$n = \frac{\frac{D_2}{2} - \frac{D_1}{2}}{h}$$

paper layers. We want to compute the length of a single paper roll; thus, we have to calculate the sum of circumferences of those layers. The sum has the following form

$$l = \sum_{i=1}^{n} 2\pi \left(\frac{D_1}{2} + ih\right) = 2n\pi\frac{D_1}{2} + 2\pi h \sum_{i=1}^{n} i = n\pi D_1 + 2\pi h \frac{(n+1)\,n}{2}\,,$$

where l is the total length of a unwound single paper roll. If we substitute for n, we get

$$l = \frac{\pi D_1 (D_2 - D_1)}{2h} + \frac{\pi (D_2 - D_1)}{2}\left(\frac{D_2 - D_1}{2h} + 1\right)\,,$$

and after the simplification

$$l = \frac{\pi}{4h}(D_2 - D_1)(D_1 + D_2 + 2h)\,.$$

Now, get back to the cylindrical shape of the toilet paper roll. The area of one unwound toilet paper is $S_0 = v \cdot l$, while the Earth's surface area is $S = 4\pi R_{\oplus}^2$. Hence, the minimum number of toilet paper rolls needed to cover the entire Earth's surface is

$$N = \frac{S}{S_0} = \frac{4\pi R_{\oplus}^2}{vl} = \frac{16hR_{\oplus}^2}{v(D_2 - D_1)(D_2 + D_1 + 2h)} \doteq 2.5 \cdot 10^{14} \text{ toilet paper rolls}\,.$$

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Problem AH ... time to shine

Neuron Endowment Fund supports talented Czech scientists and their promising projects. Jarda hopes to become a laureate of the prestigious Neuron Award one day for his research in the micro-world. Therefore, he works hard in his laboratory with an electron microscope of accelerating voltage $U = 1.5\,\mathrm{kV}$, with which he can see much more detail than using an optical microscope. Electrons in the microscope have a wavelength λ. Determine the ratio of the energy of photons of the same wavelength to the kinetic energy of electrons in the microscope.

Jarda thinks about problems even on New Year's Eve.

We will use the de Broglie wavelength of the electrons, λ, which is defined as

$$\lambda = \frac{h}{p}\,,$$

where h is Planck's constant and p is the momentum of the particle.

Since the product of the accelerating voltage and the charge of particles is small relative to the rest mass of electrons expressed in an electronvolt ($1.5\,\mathrm{keV} \ll 510\,\mathrm{keV}$), we do not have to use relativistic formulas. Therefore, the momentum is given as $p = mv$, where we obtain speed from the conservation of energy as

$$v = \sqrt{\frac{2Ue}{m_\mathrm{e}}}\,,$$

where m_e is the mass of an electron, and e is its charge.

Putting all of this together and substituing into the first equation, we obtain the wavelength of the electrons as

$$\lambda = \frac{h}{p} = \frac{h}{m_\mathrm{e} v} = \frac{h}{\sqrt{2m_\mathrm{e}Ue}}\,.$$

The energy of a photon of the same wavelength is

$$E_\gamma = hf = \frac{hc}{\lambda} = c\sqrt{2m_\mathrm{e}Ue}\,.$$

Plugging in the kinetic energy of electrons $E_\mathrm{e}^\mathrm{kin} = Ue$ as above, the sought ratio is

$$\frac{E_\gamma}{E_\mathrm{e}^\mathrm{kin}} = c\sqrt{\frac{2m_\mathrm{e}}{Ue}} \doteq 26.1\,.$$

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Problem BA ... nanoparticle coating

We have a $V_1 = 50\,\mathrm{m\ell}$ solution of silver nanoparticles with molar concentration $c_1 = 9\cdot 10^{-9}\,\mathrm{mol\cdot\ell^{-1}}$. The nanoparticles have a spherical shape with diameter $d = 20\,\mathrm{nm}$ and they are not agglomerated. We add to the beaker $V_2 = 20\,\mathrm{m\ell}$ of glucose solution with concentration $c_2 = 6.5\cdot 10^{-5}\,\mathrm{mol\cdot\ell^{-1}}$. Assume that all added glucose molecules are adsorbed onto the surface of nanoparticles and are evenly distributed on them. What will be the average number of glucose molecules per $1\,\mathrm{nm^2}$ of surface of silver nanoparticles? *Danka was inspired by her research.*

Firstly, we determine the number of silver nanoparticles N_1 in the solution as

$$N_1 = c_1 V_1 N_\mathrm{A}\,,$$

where $N_\mathrm{A} = 6.022\cdot 10^{23}\,\mathrm{mol^{-1}}$ is the Avogadro's constant. Then, using the same formula, we calculate the number of glucose molecules N_2 that were added to the nanoparticle solution as

$$N_2 = c_2 V_2 N_\mathrm{A}\,.$$

Thus, the ratio

$$\frac{N_2}{N_1} \doteq 2\,888\,.$$

yields the average number of molecules that will be adsorbed onto one nanoparticle. In order to compute the number of molecules adsorbed per unit area, we need to calculate the surface area of one nanoparticle. We do this using the formula for the surface of a sphere

$$S_1 = 4\pi\left(\frac{d}{2}\right)^2 = \pi d^2\,.$$

Finally, we can calculate the average number of glucose molecules adsorbed onto $1\,\mathrm{nm}^2$ of a silver nanoparticle surface as

$$\sigma = \frac{N_2}{S_1 N_1} = \frac{c_2 V_2}{c_1 V_1 \pi d^2} \doteq 2.30\,\mathrm{nm}^{-2}\,.$$

Thus, $2.30\,\mathrm{nm}^{-2}$ glucose molecules will be adsorbed on average onto the surface of silver nanoparticles.

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Problem BB ... racing

What distance does a car travel when accelerating from 0 *to* $100\,\mathrm{km{\cdot}h^{-1}}$ *at its maximum possible acceleration? The coefficient of friction between the road and tires is* 0.9*. Do not consider any other drag forces. Assume that the car has a powerful engine enough to maintain the maximum possible acceleration throughout the entire acceleration period.*

Martin was nostalgic about his early years at the university.

The force delivered by the engine that accelerates the car cannot be greater than the frictional force to avoid wheel slip. From Newton's 2nd law of motion, let's express the maximum acceleration a as

$$a = \frac{F}{m} = \frac{fmg}{m} = fg\,,$$

where f is the coefficient of friction. Now from the relations for velocity and the trajectory of uniformly accelerated motion, we can determinate

$$v = at = fgt \quad \Rightarrow \quad t = \frac{v}{fg}\,,$$

$$s = \frac{1}{2}at^2 = \frac{v^2}{2fg} \doteq 43.7\,\mathrm{m}\,.$$

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Problem BC ... aerogel on the water

Assume we have an aerogel cuboid with density $2.42\,\mathrm{kg{\cdot}m^{-3}}$*. We place the aerogel on the water at standard conditions. How much of the aerogel will be under the water surface? For simplicity, assume that the aerogel cannot absorb water and neglect the surface tension of the water.*

Karel wanted a problem with aerogel.

At first, note that the assumptions are not really realistic as the surface tension would be significant in this setup. We could expect that aerogel would not break the water surface, but it would descend by the same height as we determine here.

For simplicity, we can assume a cuboid, which is stable, partially submerged, and its base of area S is parallel to the water surface. The solution for a general shape of aerogel would be the same, but this approach helps us to imagine the problem better. Let us denote the depth of the submerged part of the cuboid as x, and the height of the cuboid in direction perpendicular to the water surface as h. The density of the aerogel will be ρ_1.

The buoyant force (acting on cuboid) exerted by water is $F_1 = \rho_3 Sxg$, where $\rho_3 = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water. We also have to take into account the buoyant force exerted by air, which is $F_2 = \rho_2 S\left(h - x\right) g$, where $\rho_2 = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ is the density of air. Thus, the total buoyant force acting on the aerogel is

$$F = F_1 + F_2 = S\left(\rho_3 x + \rho_2\left(h - x\right)\right) g\,.$$

The magnitude of this force has to be equal to the magnitude of the gravity acting on the aerogel, i.e., $F_g = mg = \rho_1 Shg$, from which we obtain a single equation for one unknown variable x. More precisely, we want to evaluate the ratio x/h, as it is questioned in the problem

$$F_g = F \qquad \Rightarrow \qquad \rho Shg = S\left(\rho_3 x + \rho_2\left(h - x\right)\right) g\,,$$
$$\rho_1 h = \rho_3 x + \rho_2 h - \rho_2 x\,,$$
$$\frac{x}{h} = \frac{\rho_1 - \rho_2}{\rho_3 - \rho_2} \doteq 0.122\,\%\,.$$

When we place such an aerogel on water surface without assuming surface tension, it will not descend by more than 0.13 %. In reality, aerogel would lie on the surface without breaking the water surface, and the observer would see almost no change in curvature of the water surface.

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Problem BD ... measuring the Earth radius on budget

Suppose someone lies down on the edge of the beach so that his eyes are exactly at the sea level and that he is looking at the setting sun. When the upper edge of the Sun disappears behind the horizon, he starts his stopwatch and stands up, so that his eyes are suddenly at the height of $h = 164\,\mathrm{cm}$ *and he sees the Sun again. When the upper edge of the Sun sets behind the horizon again, he stops the stopwatch. What time will the stopwatch show if it took place at the equator on the day of the equinox?* *Lego has actually never been to the seaside.*

In an Earth's reference frame, the upper edge of the Sun disk is a point which retains a circular motion with a large radius (we cannot find our distance from the Sun easily) and an angular velocity $\omega = \frac{2\pi}{24\,\mathrm{h}} \doteq 7.3 \cdot 10^{-5}\,\mathrm{s^{-1}}$. We can determine the elapsed time if we find out the angle of the point's displacement.

If the disk's upper edge disappears behind the horizon, the the line connecting their eyes and the horizon is exactly the tangent line to the Earth's surface. We will focus on two of these. The first one touches the Earth's surface at the observer's location and the second one somewhere in front of him. The angle of Sun's movement matches the included angle of these two lines. Because these tangent lines are always perpendicular to the Earth's radius at the tangent point of tangency, the angle between the lines connecting the center of Earth and the points of tangency is the same. To obtain this angle, let us assume a triangle with vertices in Earth center, the second point of tangency and the standing observer's eyes. It is obvious, that the angle in the second point of tangency is a right angle (one side belonging to this vertex is a part of the tangent line, the second is Earth's radius).

To obtain the angle in the center of Earth, we can use a trigonometric function, because we know the lengths of the sides belonging to this vertex. The adjacent side is the length between the center of Earth to the surface, which is equal to a radius R, and the hypotenuse is the radius plus the height to the observer's eyes $R + h$. The cosine is defined as a ratio of adjacent over hypotenuse, so we get the desired angle by using inverse cosine

$$\varphi = \arccos\left(\frac{R_\oplus}{R_\oplus + h}\right) \doteq 7.2 \cdot 10^{-4}\,\mathrm{rad}\,.$$

Finally, the time measured on the stopwatch is $t = \frac{\varphi}{\omega} \doteq 9.9\,\mathrm{s}$.

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Problem BE ... from extreme to extreme

What would be the length of a cylindrical rod with a radius equal to Planck length l_p and the volume of the observable universe? Assume that the universe's age is $t = 13.8 \cdot 10^9$ years and that the universe is flat. Planck length is only dependent on three fundamental physical constants – the gravitational constant G, the reduced Planck constant $\hbar$, and the speed of light c. Provide the answer in the form of $\log_{10}\left(\frac{l}{1\,\mathrm{m}}\right)$. *Robo was wondering what if...*

Firstly, we compute the Planck length. The problem statement suggests we need to use dimensional analysis to determine coefficients α, β, γ in equation $l_\mathrm{p} = G^\alpha \hbar^\beta c^\gamma$, where all the physical quantities can be expressed in SI units followingly

$$[G] = \mathrm{m}^3{\cdot}\mathrm{kg}^{-1}{\cdot}\mathrm{s}^{-2}\,, \qquad [c] = \mathrm{m}{\cdot}\mathrm{s}^{-1}\,,$$
$$[\hbar] = \mathrm{J}{\cdot}\mathrm{s} = \mathrm{m}^2{\cdot}\mathrm{kg}{\cdot}\mathrm{s}^{-1}\,, \quad [l_\mathrm{p}] = \mathrm{m}\,.$$

Substituting these expressions into the equation for l_p, we get

$$\mathrm{m}^1 = \mathrm{m}^{3\alpha} \cdot \mathrm{kg}^{-\alpha} \cdot \mathrm{s}^{-2\alpha} \cdot \mathrm{m}^{2\beta} \cdot \mathrm{kg}^{\beta}\mathrm{s}^{-\beta} \cdot \mathrm{m}^{\gamma} \cdot \mathrm{s}^{-\gamma}\,.$$

By comparing the bases of powers on the right and left side of the equation, we get three equations with three unknowns

$$1 = 3\alpha + 2\beta + \gamma\,,$$
$$0 = -\alpha + \beta\,,$$
$$0 = -2\alpha - \beta - \gamma\,.$$

From the second equation, we know that $\beta = \alpha$, consequently from the third equation, we get $\gamma = -3\alpha$, and lastly we substitute into the first equation to obtain $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, and $\gamma = -\frac{3}{2}$. Thus, Planck length can be expressed as

$$l_\mathrm{p} = G^{\frac{1}{2}} \cdot \hbar^{\frac{1}{2}} \cdot c^{-\frac{3}{2}} = \sqrt{\frac{G\hbar}{c^3}} = 1.616 \cdot 10^{-35}\,\mathrm{m}\,.$$

The radius of the observable universe is $R = tc$, expressed numerically (in meters) as

$$R = 13.8 \cdot 10^9 \doteq 1.305\,6 \cdot 10^{26}\,\mathrm{m}\,.$$

We proceed by setting the volumes to be equal,

$$S \cdot l = \frac{4}{3}\pi R^3\,,$$

where $S = \pi l_\mathrm{p}^2$. The length of the rod is finally

$$l = \frac{4R^3}{3l_\mathrm{p}^2} = \frac{4 \cdot 2.23 \cdot 10^{78}\,\mathrm{m}^3}{3 \cdot 2.61 \cdot 10^{-70}\,\mathrm{m}^2} \doteq 1.1 \cdot 10^{148}\,\mathrm{m}\,.$$

Hence, the correct answer is 148.

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Problem BF ... little wrecking ball

We are at Jupiter's surface, where the gravity is $a = 24.8\,\mathrm{m{\cdot}s^{-2}}$ and we have two identical pendulums, which oscillate in one plane and share the same hanging point. The length of rigid massless string is $l = 5\,\mathrm{m}$, and the ball made of osmium at its end has diameter $d = 5\,\mathrm{cm}$. Each of pendulums has an initial amplitude of $\theta = 4^\circ$. Consequently, we let them swing towards each other and eventually collide.

Now we assume only one of the pendulums. At what velocity does the ball need to collide with the rigid vertical wall (perpendicular to the plane of oscillations and containing the hanging point and the equilibrium position), so that ball will experience the same force as in the case of the two-pendulum collision mentioned above? All collisions assume to be perfectly elastic.

Delion was thinking 'bout traffic accidents.

We approach the problem using the law of conservation of total energy. At first, we solve the case of two pendulums colliding with each other. Just before the collision, each of the pendulums has its maximum velocity v_max, which they obtained by conversion of potential energy E_p to kinetic energy E_k. The potential energy E_p of one pendulum is

$$E_\mathrm{p} = mah\,,$$

where m is the ball's mass, and h is the height of the ball's initial position with respect to the equilibrium position. Concerning the size of the ball and the characteristic dimensions of the problem, we can approximate the ball by a mass point at its center. Thus, the height of the initial position h is

$$h = \left(l + \frac{d}{2}\right)(1 - \cos\theta)\,.$$

Kinetic energy E_k of the ball is

$$E_\mathrm{k} = \frac{1}{2}mv^2\,.$$

If we substitute for h in the potential energy formula, and set the potential and kinetic energy equal, we get

$$ma\left(l + \frac{d}{2}\right)(1 - \cos\theta) = \frac{1}{2}mv_\mathrm{max}^2\,.$$

We simplify the equation for v_{max} and obtain the formula for ball's velocity just before the collision

$$v_{\mathrm{max}} = \sqrt{2a\left(l + \frac{d}{2}\right)(1 - \cos\theta)}\,.$$

The case with two pendulums and the questioned case (a pendulum and a rigid wall) are the equivalent if we look only at the half-plane, where the pendulum is in both cases. Therefore, we only need plug in the numbers to the last equation, obtaining $v_{\mathrm{max}} \doteq 0.78\,\mathrm{m{\cdot}s^{-1}}$. Concerning the dimensions of the ball at the end of the string, we neglected its moment of inertia.

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Problem BG ... traffic density on a highway

Dano drives on the highway and sees trucks in the opposite direction more often than those he overtakes. He wonders whether this is correct and if there is not just more traffic in the opposite direction than he is driving. What should be the ratio of the number of trucks he passes in the opposite direction to those he overtakes if there is the same number of trucks per unit time in both directions? For simplicity assume that Dano is driving at speed $v_1 = 130\,\mathrm{km{\cdot}h^{-1}}$, and all trucks drive at speed $v_2 = 90\,\mathrm{km{\cdot}h^{-1}}$. You can neglect the dimensions of cars, trucks, and highway. *Karel was driving the highway for several hours and has been thinking about it.*

Dano passes trucks in the opposite direction at speed $w_1 = v_1 + v_2$ because the speeds of vehicles going in opposite directions add up. On the contrary, he overtakes the trucks at speed $w_2 = v_1 - v_2$. If we observe a long period, which means many trucks, or if we have no information about spacings, it should be obvious that we must solve directly by using speeds. The faster the trucks move towards Dano's car, the more of them he meets in a certain period of time. The questioned ratio K is thus directly proportional to the ratio of speeds by which he passes the trucks; we can write

$$K = \frac{w_1}{w_2} = \frac{v_1 + v_2}{v_1 - v_2} = \frac{220}{40} = 5.5\,,$$

If the truck traffic was the same in both directions, then Dano should pass the oncoming trucks 5.5-times more often than those he overtakes. But if he passes, e.g., only 4-times more, he can assume that the traffic in the opposite direction is lower, although he sees the oncoming trucks more often.

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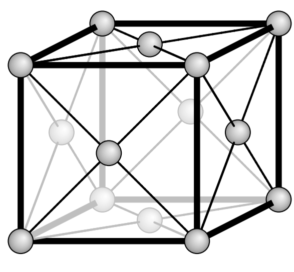
Problem BH ... dense lattice

One of the possible arrangements of atoms in the crystal is the face-centered cubic lattice. In this case, the atoms are located at the vertices and centers of the walls of the cube. The crystal is formed by the periodic arrangement of these cubes. We consider atoms to be rigid spheres that come in close contact. What part of the crystal volume is filled with atoms?

Danka studied crystallography.



Let us denote the length of the elementary cell edge as a. The first important thing is to determine the maximum possible radius r of the atom in the lattice. Let us look at how the atoms are arranged in the cube's wall. There is a single atom in each vertex, and another atom lies in the center of the wall. Since the atoms are in close contact, the length of the wall diagonal is four times the atom's radius (two contributions from the center atom and one from each of the two atoms at opposite vertices). The length of the wall diagonal is a hypotenuse of a right triangle with perpendiculars of length a. We find that the wall diagonal has the length $\sqrt{2}a$. It holds

$$\sqrt{2}a = 4r\,,$$

and thus

$$r = \frac{a}{2\sqrt{2}}\,.$$

We also have to determine how many atoms belong to one elementary cell. Each atom at the vertex of this imaginary cube in the crystal lattice belongs to eight adjacent cubes, and thus only the $\frac{1}{8}$ of the vertex atom belongs to our elementary cell. Since the cube has 8 vertices, the total volume of the vertex atoms corresponds to $8 \cdot \frac{1}{8} = 1$ whole atom. Analogously, each atom in the middle of the wall belongs to two elementary cells, so there is only $\frac{1}{2}$ wall atom per cube. The cube has 6 walls, so the total volume of wall atoms in one unit cell is then equal to $6 \cdot \frac{1}{2} = 3$ whole atoms. Thus, a space of four whole atoms is filled in one unit cell. Finally, calculate the ratio p of the total volume of atoms in the unit cell to the volume of the cell itself

$$p = \frac{4\frac{4}{3}\pi r^3}{a^3}\,.$$

This may be recasted using the relation between the radius and the length of the elementary cell edge as

$$p = \frac{\pi}{3\sqrt{2}} \doteq 74\%\,.$$

The filling factor of the face-centered cubic (FCC) grid is, therefore, 74 %.

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Problem CA ... arachnophobia

Lying on his bed, Jarda was watching a spider hanging from the ceiling on its spider silk. Jarda thought to himself how lucky he was that spiders are not bigger. However, Jarda got an idea that perhaps, spiders could not be bigger by much. If spider silk can hold a k-multiple of spider's mass, what is the maximum factor by which a spider and spider silk could enlarge (while maintaining all the proportions) so that spider silk does not break?

Jarda was watching a horror movie.

The information that tells us how much spider silk can carry is the ultimate tensile stress that spider silk can withstand. The stress depends on the acting force and the area of the silk's cross-section. If a spider gets bigger by a factor c, its mass increases proportionally to c^3, while the cross-section of silk increases proportionally to c^2. Thus, the maximal acting force increases proportionally to c^2. The ultimate tensile strength of spider silk is kmg; thus, concerning the condition for non-rupture of silk, we get

$$c^3mg \leq c^2kmg\,,$$

clearly, the maximum factor is $c_{\max} = k$.

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Problem CB ... eighth-life

What is the time required to decay one-eighth of unstable radioactive particles if we know that half of the particles decays in time T? *Karel wanted to trap naive participants.*

Radioactive decay can be expressed by the following formula

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{T}}\,,$$

where $N(t)$ is the number of particles that have not undergone decay yet at time t, and N_0 is the initial total number of the particles. We express the ratio of number of particles that have not udergone decay yet to the initial number of particles as

$$\frac{N(t)}{N_0} = \left(\frac{1}{2}\right)^{\frac{t}{T}}\,.$$

Now we have to realize that we are looking for the time at which $1 - \frac{1}{8} = \frac{7}{8} = 87.5\,\%$ of the original particles are still left, thus

$$\left(\frac{1}{2}\right)^{\frac{t}{T}} = \frac{7}{8} \quad \Rightarrow \quad \frac{t}{T}\ln\frac{1}{2} = \ln\frac{7}{8} \quad \Rightarrow \quad t = \frac{\ln\frac{7}{8}}{\ln\frac{1}{2}}T \doteq 0.193T\,.$$

One-eighth of the unstable particles decays in 0.193-multiple of half-life. It is a little quicker than what “a premature estimate,” that it is one-quarter of half-life could be. The closer to the beginning, the more original particles we have, and thus the more frequently decay occurs.

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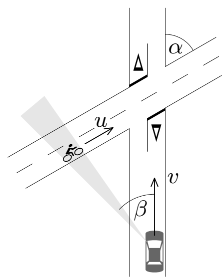
Problem CC ... attention, cyclist

A car is approaching an intersection on a straight road at speed $v = 50\,\mathrm{km\cdot h^{-1}}$. At the intersection, the car should give a right of way to the vehicles going at the other straight road, which crosses the first road from the left at angle $\alpha = 60^\circ$. The driver's view is obstructed by the front left pillar of the car bodywork at angle $\beta = 25^\circ$ (from the direction of the travel). At what speed u must the cyclist approach the intersection in order to collide with the car without being seen by the driver?

Dodo does not ride a bicycle much.



Let t be the time remaining until the collision. Since the car and
cyclist should collide at the intersection, the car must be at distance vt from it, and the cyclist at distance ut, at each time. To prevent the driver from seeing the cyclist, the positions of cyclist C, car A, and intersection K must form a triangle with angles β opposite to side CK, α opposite to side AC, and $\gamma = \pi - \alpha - \beta = 95^\circ$ opposite to side AK. From the knowledge of the angles and the length of side AK, we can use the law of sines to express side KC as

$$|KC| = |AK| \frac{\sin\beta}{\sin\gamma} .$$

Then we obtain the speed of the cyclist as

$$u = v\frac{\sin\beta}{\sin\gamma} = v\frac{\sin\beta}{\sin(\alpha+\beta)} \doteq 21\,\mathrm{km{\cdot}h^{-1}} .$$

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Problem CD ... neglecting Saturn

The organizers of FYKOS plan to laser evaporate Saturn and then homogenize its rings into a hoop of a radius r and length density τ. Determine the gravitational force that would act on us in the center of the hoop. Consider that the organizers were also careful to evaporate the Saturn moons. Neglect the influences of the surrounding planets. Do not question the motives of the organizers. *Pepa wants to be fired from FYKOS.*

Concerning the problem symmetry, one may intuitively expect that the force in question will be zero – the gravitational effects of the opposite elements of the hoop will cancel each other out. We can verify this assumption by the following calculation.

Working in the plane of the hoop, we introduce the polar coordinates with the origin at its center. The position of any point of the hoop is thus described using a position vector

$$\mathrm{d}\mathbf{r} = (r\cos\varphi, r\sin\varphi) .$$

Using the length density, we can express mass of an infinitesimal section of the hoop as $\mathrm{d}m = \tau\,\mathrm{d}l$, or in our coordinate system as $\mathrm{d}m = \tau r\,\mathrm{d}\varphi$. Its contribution to the net force acting on object of a mass M at its center is then calculated using the Newton's law as

$$\mathrm{d}\mathbf{F} = G\frac{M\,\mathrm{d}m}{r^2}\frac{\mathrm{d}\mathbf{r}}{r}.$$

The resulting force is obtained by summing the individual contributions from all "pieces" of the hoop. However, we must not forget that we add up vectors. We can divide the integral into two components

$$\int_{\mathrm{hoop}} \mathrm{d}\mathbf{F}_{\mathrm{x}} = \frac{GM\tau}{r^2}\int_0^{2\pi}\cos\varphi\,\mathrm{d}\varphi = 0\,,$$

$$\int_{\mathrm{hoop}} \mathrm{d}\mathbf{F}_{\mathrm{y}} = \frac{GM\tau}{r^2}\int_0^{2\pi}\sin\varphi\,\mathrm{d}\varphi = 0\,.$$

The contributions from the individual sections of the hoop cancel each other out, and thus net force acting on the object in the center of the hoop is indeed zero.

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Problem CE ... heating of the synthetic air

Consider an insulated vacuum chamber with dimensions $a = 5\,\mathrm{cm}$, $b = c = 4\,\mathrm{cm}$ filled with nitrogen and oxygen at the temperature $20\,\mathrm{^\circ C}$ in a molar ratio of 85 to 15 so that the pressure in the chamber is $100\,\mathrm{mbar}$. We'll start heating the gas mixture at a rate of $0.8\,\mathrm{^\circ C{\cdot}s^{-1}}$. What will be the power input to the chamber if only 68 % of used energy is consumed for gas heating? The specific heat capacity of nitrogen is $c_{\mathrm{N_2}} = 743\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ and for oxygen it's $c_{\mathrm{O_2}} = 658\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. The molar masses of nitrogen and oxygen are $M_{\mathrm{N_2}} = 28\,\mathrm{g{\cdot}mol^{-1}}$ and $M_{\mathrm{O_2}} = 32\,\mathrm{g{\cdot}mol^{-1}}$ respectively. *Danka studied operando methods.*

For the heat Q received by a body, we can write a calorimetric equation in the form

$$Q = mc\Delta T\,,$$

where m is the body's mass, c is its specific heat capacity and ΔT is the change in the body temperature due to received heat. Thus, we can write an equation for a given gas mixture

$$đQ = \left(m_{\mathrm{N_2}} c_{\mathrm{N_2}} + m_{\mathrm{O_2}} c_{\mathrm{O_2}}\right) \mathrm{d}T\,,$$

where $đQ$ is the infinitesimal amount of heat supplied to the mixture, and $\mathrm{d}T$ is the infinitesimal increase in gas temperature. Additionally, the supplied heat is related to the input power P as $đQ = \eta P\,\mathrm{d}t$, where η is the efficiency of the gas heating.

We can determine the amount of substance in the chamber using the equation of the state of an ideal gas

$$pV = nRT\,,$$

where p is the gas pressure, V its volume, n the amount of substance, R the molar gas constant, and T the gas temperature. Since the amount of substance is constant throughout the whole process, we can determine it using the initial values as

$$n = \frac{p_0 V}{RT_0}\,.$$

This equation applies to each gas separately. The volume of both gases is equal to the chamber volume, i.e., $V = abc$. According to Dalton's law, the total pressure of a gas mixture is equal to the sum of the partial pressures of the individual components. At the same time, the ratio of the partial pressures of the individual gases is equal to their molar ratio. It follows that the initial partial pressure of nitrogen in the chamber is $p_{\mathrm{N_2}} = 85\,\mathrm{mbar}$ and oxygen $p_{\mathrm{O_2}} = 15\,\mathrm{mbar}$. The amount of substance can be easily converted to the weight of the gas using the relation

$$m = nM\,,$$

where M is the molar mass of the gas. Combining the relations mentioned above we get

$$\eta P\,\mathrm{d}t = \frac{V}{RT_0}\left(p_{\mathrm{N_2}} M_{\mathrm{N_2}} c_{\mathrm{N_2}} + p_{\mathrm{O_2}} M_{\mathrm{O_2}} c_{\mathrm{O_2}}\right) \mathrm{d}T\,,$$

$$P = \frac{abc}{RT_0\eta}\left(p_{\mathrm{N_2}} M_{\mathrm{N_2}} c_{\mathrm{N_2}} + p_{\mathrm{O_2}} M_{\mathrm{O_2}} c_{\mathrm{O_2}}\right) \frac{\mathrm{d}T}{\mathrm{d}t}\,.$$

The expression $\frac{\mathrm{d}T}{\mathrm{d}t}$ is the rate of change of the mixture temperature. We are left to substitute numerical values into the formula. By doing so we get $P = 8.05 \cdot 10^{-3}\,\mathrm{W}$. Thus, the power input is $P = 8.05\,\mathrm{mW}$.

We can obtain a similar result in a theoretical way - from the ideal gas model. Since no work is done on the system (its volume does not change, and there is no chemical reaction), the 1st law of thermodynamics takes the form

$$\mathrm{d}U = đQ\,,$$

where U is the internal energy, which in the case of a diatomic gas is $U = \frac{5}{2}nRT$. Pluging this into a aforementioned relationship $\eta P\,\mathrm{d}t = đQ$, one obtains

$$\eta P = \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{5}{2}nR\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{5}{2}\frac{p_0 V}{T_0}\frac{\mathrm{d}T}{\mathrm{d}t}\,.$$

In this way we get the value $P = 8.03\,\mathrm{mW}$. The different result is caused by the difference between the theoretically and experimentally determined value of specific heat capacity. We can determine its value in our microscopic gas model as

$$mc\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{5}{2}nR\frac{\mathrm{d}T}{\mathrm{d}t}\,,$$

from which it follows that

$$c = \frac{5}{2}\frac{nR}{m} = \frac{5}{2}\frac{R}{M}\,.$$

Compared to the exerimental values, we get $c_{N_2} = 742\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ for nitrogen and $c_{O_2} = 650\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ for oxygen.

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Problem CF ... marble is jumping off the window

A marble (a small hard ball) with mass $1.2\,\mathrm{kg}$ *fastens itself to an elastic rope with stiffness* $6\,\mathrm{N{\cdot}m^{-1}}$ *and free length* $l_0 = 8\,\mathrm{m}$*. The other end of the rope is attached to a pad from which the marble jumps off. The rope unwinds during the fall and starts to extend when the marble is at distance* l_0 *below the pad. The marble is afraid of dizziness, so it wants to know the maximum acceleration (g-force) it faces during its journey? Write your answer a multiple of* g*.* *Jarda will not try this at the dormitory.*

During the fall, until the whole rope unwinds and straightens, the marble moves with acceleration g. When the rope is extended, the marble will start moving like a harmonic oscillator. For the harmonic oscillator, the greatest acceleration is at the maximum amplitude of the motion, and the speed there is zero. Thus, the maximum extension of the rope can be found using the law of conservation of mechanical energy, expressed as

$$mg\,(l_0 + y) = \frac{1}{2}ky^2\,,$$

where m and k are the mass of the marble and the stiffness of the rope. The left-hand side represents the decline of potential energy, the right-hand side is the elasticity energy gain, and y denotes the maximum extension. We express y from the quadratic equation as

$$y = \frac{mg \pm \sqrt{m^2g^2 + 2mgl_0k}}{k}\,,$$

while we seek the positive root. Therefore, the magnitude of the force acting on the marble at the lowest point is

$$F = ky - mg = \sqrt{m^2g^2 + 2mgl_0k}\,,$$

and hence, the acceleration here is

$$a = \frac{F}{m} = \sqrt{g^2 + \frac{2gl_0k}{m}}\,.$$

We find the g-force as the ratio of the final acceleration and the gravitational acceleration

$$\frac{a}{g} = \sqrt{1 + \frac{2l_0k}{mg}} \doteq 3.0\,.$$

Thus, the maximum g-force acting on the marble is three times the gravitational acceleration.

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Problem CG ... thermos suction cup

Legolas noticed that when rinsing his thermos with cold water, covering the opening with his hand and shaking the thermos, there is a negative pressure forming inside the thermos, sometimes sufficiently large to keep the thermos hanging on his hand. He assumes that this happens because the air in the thermos was initially warm from the tea he had had in the bottle before, and the cold water quickly cools the air inside. So, he decided to make a problem with similar theme.

Consider now a thermos of volume $V = 500\,\mathrm{m\ell}$ *and opening area* $S = 13\,\mathrm{cm^2}$*. Mass of the water and the thermos in total is* $m = 0.35\,\mathrm{kg}$ *and the temperature of the air inside equilibrates to* $T_\mathrm{c} = 20\,\mathrm{^\circ C}$ *after rinsing. What is the (minimal) original temperature of air inside the thermos, so that the created negative pressure keeps the thermos stuck to Lego's hand?* *True story.*

Before covering the opening, the air in the thermos had temperature T_h and was at atmospheric pressure $p_\mathrm{h} = p_\mathrm{a}$. The whole process is isochoric – the volume of the system is constant, hence after cooling of the air to temperature $T_\mathrm{c} = 293.15\,\mathrm{K}$, the equilibrium pressure of the air inside the thermos is $p_\mathrm{c} = p_\mathrm{h}T_\mathrm{c}/T_\mathrm{h}$. The magnitude of the force that pushes the thermos against the hand is given by the product of the opening area and the pressure difference

$$F = S\Delta p = Sp_\mathrm{c}\left(1 - \frac{T_\mathrm{c}}{T_\mathrm{h}}\right) = Sp_\mathrm{c}\frac{T_\mathrm{h} - T_\mathrm{c}}{T_\mathrm{h}}\,.$$

In the opposite direction, the weight of the system pulls on the thermos, leading to an equilibrium condition for minimum initial temperature

$$mg = Sp_\mathrm{c}\frac{T_\mathrm{h} - T_\mathrm{c}}{T_\mathrm{h}} \quad \Rightarrow \quad T_\mathrm{h} = T_\mathrm{c}\frac{Sp_\mathrm{c}}{Sp_\mathrm{c} - mg} \doteq 301\,\mathrm{K} = 28\,\mathrm{^\circ C}\,.$$

This is a relatively low temperature considering that the air was heated by contact with tea. That would suggest that the thermos sticks to Lego's hand quite often, but he denies that. The discrepancy is probably caused by the various approximations we made during the calculation – in a real situation, the air begins to cool down already during the process of pouring the water into the thermos.

Also, Legolas' hand is not perfectly insulating, so some air slips into the thermos even after the "sealing". Furthermore, Legolas' hand is not perfectly stiff, so it changes volume when compressed by the surrounding air, and hence changes the volume of the air inside the system.

For more experimentally inclined problem solvers, Legolas claims absolutely no responsibility for thermos destroyed as a result of this problem.

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Problem CH ... Robo shoot!

Robo is at the distance $L = 2\,022\,\mathrm{cm}$ from a soccer goal and is kicking a ball towards it with initial speed $v_0 = 15\,\mathrm{m{\cdot}s^{-1}}$ at angle $\alpha = 45^\circ$. Robo hits the crossbar, and unexpectedly, the ball bounces off perfectly vertically upwards but loses $10\,\%$ of its velocity due to the bounce. What maximum height above ground does the ball reach after it bounces upwards?

Robo likes to hit the construction of a soccer goal.

We use the motion equations of a upward parabolic throw

$$x(t) = v_0 t\cos\alpha\,, \qquad v_x(t) = v_0\cos\alpha\,,$$
$$y(t) = v_0 t\sin\alpha - \frac{1}{2}gt^2\,, \quad v_y(t) = v_0\sin\alpha - gt\,,$$
$$v = \sqrt{v_x^2 + v_y^2}\,.$$

where $x(t)$ and $y(t)$ are the instantaneous coordinates of the ball in time t, $v_x(t)$ and $v_y(t)$ are components of the velocity v. The ball will be in horizontal distance $x(t) = L$, at time $t_1 = \frac{L}{v_0\cos\alpha}$ and its height above the ground will be the same as the height of the soccer goal $y(t_1) = h_0$. The height of the soccer goal can be obtained by plugging the time t_1 into the second equation of parabolic throw. Consequently we get

$$h_0 = v_0\frac{L}{v_0\cos\alpha}\sin\alpha - \frac{1}{2}g\left(\frac{L}{v_0\cos\alpha}\right)^2 = L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\,.$$

Now, we should take a look on the velocity after bounce. Let us denote the "reduction" factor as $\gamma = 0.9$, then

$$v_2 = \gamma v_1 = \gamma\sqrt{v_0^2\cos^2\alpha + \left(v_0\sin\alpha - \frac{gL}{v_0\cos\alpha}\right)^2} = \gamma\sqrt{v_0^2 - 2g\left(L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\right)}\,,$$

where v_1 is the ball's velocity before the bounce, and v_2 is the ball's velocity after the bounce. Consequently, we can solve the problem using the law of conservation of energy. We take into account two situations – the situation right after the bounce from the crossbar and then the situation when the ball is at its maximum height after the bounce and the ball's velocity is therefore zero. This is represented by the following set of equations

$$\frac{1}{2}mv_2^2 + mgh_0 = mgh_{\max} \quad \Rightarrow \quad h_{\max} = h_0 + \frac{v_2^2}{2g}\,.$$

We substitute for h_0 and v_1 from the equations above, and we finally get the solution

$$h_{\max} = L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha} + \frac{\gamma^2}{2g}\left(v_0^2 - 2g\left(L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\right)\right)\,,$$

$$h_{\max} = \left(1-\gamma^2\right)\left(L\tan\alpha - \frac{gL^2}{2v_0^2\cos^2\alpha}\right) + \gamma^2\frac{v_0^2}{2g} \doteq 9.74\,\mathrm{m}\,.$$

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Problem DA ... waterboarding of iron ball

Legolas took his aquarium, filled it with water (but not completely to the edge, not to overfill it), and placed it on a weighing scale. Legolas did not have any fish, so he decided to float his favorite iron ball there. Of course, the ball alone would immediately sink, so he hung it on a gauge meter.

When the ball was floating under the surface (completely submerged, but not touching either the bottom, or the water level), the gauge meter displayed value $\Delta F = 10\,\mathrm{N}$. Nevertheless, Lego noticed that the weighing scale now displayed a greater value than before submerging the iron ball.

By how much did the number displayed on the weighing scale increase? The density of iron is $\rho_{\mathrm{Fe}} = 7\,874\,\mathrm{kg{\cdot}m^{-3}}$. Lego does not know the exact dimensions of his iron ball. Know that no water leaked from the aquarium, and that the weighing scale operates in kilograms.

Legolas has borrowed...

The first and most important is to realize that Newton's third law always applies. Indeed, if water lightens the iron ball by buoyancy force, the ball presses on the water, so its total force has the same magnitude as the buoyancy force and points downwards. The difference in displayed values reflects the buoyancy force acting on the iron ball.

Since the weighing scale converts the difference in forces to weight, we can say that the displayed value will increase by the weight of uplifted water. We only need to find the volume of the iron ball. Furthermore, we know that if the ball is fully submerged, the gauge meter has to compensate for the difference between gravity and buoyancy force by force of magnitude ΔF. This provides us with the equality

$$\Delta F = F_g - F_{\mathrm{b}} = V\rho_{\mathrm{Fe}}g - V\rho_{\mathrm{water}}g \quad \Rightarrow \quad V = \frac{\Delta F}{g\left(\rho_{\mathrm{Fe}} - \rho_{\mathrm{water}}\right)}\,.$$

To get the mass of uplifted water, which makes the difference between displayed values, we only need to multiply obtained volume by water density

$$\Delta m = \frac{\rho_{\mathrm{water}}}{\rho_{\mathrm{Fe}} - \rho_{\mathrm{water}}}\frac{\Delta F}{g} \doteq 0.15\,\mathrm{kg}\,.$$

If the weighing scale operates in kilograms, the displayed value increased by 0.15.

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Problem DB ... repulsive steering wheel

We choose two points on a circumference made of conductive material with specific linear resistance λ. What is the inscribed angle between these two points so that the resistance between them is half the maximum resistance possible (assuming a different position of the points)? Provide the value less than π as the answer. *Jarda comes up with problems when driving.*

We choose an arbitrary point on a circumference, and then we (without loss of generality) search for a second point, which is angularly distant φ ($2\pi - \varphi$, respectively) from the first one. The resistance between these two points is equivalent to the parallel setup of two resistors with resistance $\varphi\lambda r$ and $(2\pi - \varphi)\,\lambda r$, where r is the radius of the circumference. Total resistance is indeed

$$R(\varphi) = \frac{(2\pi - \varphi)\,\varphi\lambda^2 r^2}{2\pi\lambda r} = \frac{(2\pi - \varphi)\,\varphi\lambda r}{2\pi}\,.$$

The denominator is not dependent on φ, to obtain the maximal resistance, we differentiate the numerator

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\,(2\pi - \varphi)\,\varphi = 2\pi - 2\varphi\,.$$

Setting this equal to zero, we find that the angle for which the resistance reaches its maximum (due to the negative second derivative) is $\varphi_{\mathrm{max}} = \pi$. Unsurprisingly, we could have expected this result thanks to the symmetry of the problem. Therefore, the maximum resistance possible is

$$R_{\mathrm{max}} = \frac{(2\pi - \pi)\,\pi\lambda r}{2\pi} = \frac{\pi\lambda r}{2}\,.$$

We are asked to find such an angle φ, for which $R(\varphi) = \frac{R_{\mathrm{max}}}{2}$. To get the answer, we have to solve the following equation

$$\frac{\pi\lambda r}{4} = \frac{(2\pi - \varphi)\,\varphi\lambda r}{2\pi}\,.$$

We simplify it to quadratic equation

$$2\varphi^2 - 4\pi\varphi + \pi^2 = 0\,,$$

whose solution is

$$\varphi_{1,2} = \frac{4\pi \pm \sqrt{16\pi^2 - 8\pi^2}}{4} = \frac{2 \pm \sqrt{2}}{2}\pi\,.$$

Both solutions are valid because their sum is 2π. Nevertheless, in the problem statement, we ask on smaller of both angles; thus, the correct answer is $\frac{2-\sqrt{2}}{2}\pi = 0.92\,\mathrm{rad}$.

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Problem DC ... current of the metro

Metro train of type M1 is $D = 96.66\,\mathrm{m}$ *long,* $w = 2.72\,\mathrm{m}$ *wide, and its mass is* $M = 133\,\mathrm{tons}$*. Its capacity is* $N = 1\,464$ *travelers, the maximum speed is* $v_{\mathrm{max}} = 90\,\mathrm{km{\cdot}h^{-1}}$*, and the acceleration is* $a = 1.4\,\mathrm{m{\cdot}s^{-2}}$*. If we assume the metro accelerates with the constant acceleration (from zero to the maximum speed), what is the maximum current that the metro train would need to take from the railways?*

Assume the metro train is fully loaded by people of average weight $m = 85\,\mathrm{kg}$*, including clothes and luggage. The voltage in the railways is* $U = 750\,\mathrm{V}$*. Furthermore, neglect resistance forces, the engine efficiency is* $80\,\%$*. The metro moves prefectly horizontally.*

Karel was thinking about power of the metro.

We approach the solution by the output power of the metro train. Since the output power of the engine is $\eta = 80\,\%$ of the input power, the following holds $P = \eta U I$. This is the output power that accelerates the train, it's instantaneous value is

$$P = Fv = M_{\mathrm{tot}} a v = (M + mN)\, av\,,$$

where F is the instantaneous force (in the direction of travel) magnitude, v is the train speed, a is the instantaneous (yet constant) acceleration, and M_{tot} is the total mass of the loaded train. Because the mass and the acceleration remain constant, the greatest power output occurs just before reaching the maximum speed (at which the train stops accelerating). If we wanted to be more realistic, we would have to get the data about the particular acceleration of the train at a certain speed. From now on, we use v_{max} instead of v.

Now, we only need to put previous formulae together and compare them

$$P = \eta U I = (M + mN)\, a v_{\mathrm{max}} \quad \Rightarrow \quad I = \frac{(M + mN)\, a v_{\mathrm{max}}}{\eta U} \doteq 15\,000\,\mathrm{A} = 15.0\,\mathrm{kA}\,.$$

If the train were accelerating by the assumptions, the maximal electric current consumption would occur just before reaching the maximum speed. The electric current value would be 15.0 kA.

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Problem DD ... fun at lecture

Jarda lost his attention at a lecture, so he started to play with his pen, which is of rest length l_0*. He put the pen vertically and pressed it by* $\Delta l \ll l_0$*. Then he released it, the spring inside the pen stretched, the pen jumped vertically upwards, and its tip reached the height* h*. Jarda had fun for a while, but he got bored quickly.*

Consequently, he dismantled the pen, removed the spring, and hung the rest of the pen on it. What was the frequency of the oscillations of a pen on the spring? The mass of the spring is much less than the mass of the rest of the pen.

Jarda is afraid that someone might have a wild imagination…

The elastic energy

$$\frac{1}{2} k\, (\Delta l)^2$$

is used to increase the potential energy by $mg\left(h-l_0\right)$, where k is the spring stiffness, m is the mass of the pen. The law of conservation of energy can be simplified and written as

$$\frac{k}{m}=\frac{2g\left(h-l_0\right)}{\left(\Delta l\right)^2}\,.$$

The frequency formula for a harmonic oscillator with parameters k and m is

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}\,.$$

We substitute for the fraction in a square-root from the law of conservation of energy, and get

$$f=\frac{1}{2\pi}\frac{\sqrt{2g\left(h-l_0\right)}}{\Delta l}\,.$$

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Problem DE ... hot plates

Danka was heating water on two hot plates simultaneously. The diameter of the large plate is $d_\mathrm{l}=19\,\mathrm{cm}$ and of the small one is $d_\mathrm{s}=15\,\mathrm{cm}$. On the large plate, $V_\mathrm{l}=2\,\ell$ of water is being heated in a pot which weights $m_\mathrm{l}=1\,\mathrm{kg}$. On the small plate, $V_\mathrm{s}=1\,\ell$ of water is being heated in a smaller pot weighting $m_\mathrm{s}=0.5\,\mathrm{kg}$. Both pots are made out of the same material with a specific heat capacity $c=450\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. Let us assume that the power $P=8\,\mathrm{kW}$ supplied to the two-plate hob is distributed between the hot plates proportionally to their area. The efficiency of each plate is $\eta=80\,\%$. Furthermore, consider that all heat given off by the cooking plates is absorbed by the respective pot and the water inside and is not distributed any further. What is the time delay between reaching the boiling point in the smaller pot and in the large one? In the beginning, the water and the pots have room temperature $T_0=20\,^\circ\mathrm{C}$.

Danka experimentally verified the functioning of her dormitory hob.

For the amount of heat supplied by the plate i in the time t_i to the respective pot filled with water, the following applies

$$P_it_i=Q_i=\left(m_ic+V_i\rho c_\mathrm{w}\right)\left(T_1-T_0\right)\,,$$

where t_i is the necessary time for the water in the pot i to start to boil, $\rho=998\,\mathrm{kg\cdot m^{-3}}$ is the density of water, $c_\mathrm{w}=4\,184\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ is water's specific heat capacity and $T_1=100\,^\circ\mathrm{C}$ is the boiling point of water. From the problem statement for the effective power of the plate i applies

$$P_i=P\eta\frac{S_i}{S_\mathrm{l}+S_\mathrm{s}}\,.$$

In the previous equation, $S_i=\pi\frac{d_i^2}{4}$ is the area of the hot plate i. From the fist equation, we can express the time it takes for the water to boil, and then we can calculate the difference $\Delta t=t_\mathrm{l}-t_\mathrm{s}$. Substituting all expressions and with appropriate adjustments we get the following equation

$$\Delta t=\frac{T_1-T_0}{P\eta}\left[\left(m_\mathrm{l}c+V_\mathrm{l}\rho c_\mathrm{w}\right)\left(1+\frac{d_\mathrm{s}^2}{d_\mathrm{l}^2}\right)-\left(m_\mathrm{s}c+V_\mathrm{s}\rho c_\mathrm{w}\right)\left(1+\frac{d_\mathrm{l}^2}{d_\mathrm{s}^2}\right)\right]\,.$$

After substituting the numerical values, we find that $\Delta t \doteq 35\,\mathrm{s}$. Therefore, the water in the large pot reaches boiling point 35 s later than the water in the smaller one.

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Problem DF ... fast particles

Very fast particles with mean lifetime $t_0 = 1.3 \cdot 10^{-6}\,\mathrm{s}$ arrive from space. In a balloon at height $h = 2\,\mathrm{km}$ above the ground, the detector measures $N_1 = 1\,100$ of them in some time. At the ground level, the same detector measures $N_2 = 170$ of them in the same time period. What is the speed of these particles? Express the result as a multiple of the speed of light c.

Danka recalled the course on special relativity.

If a particle is relativistic (moves at speed close to the speed of light), a time dilatation occurs, which means that observer at rest measures a longer lifetime of particle t than the proper lifetime of particle t_0. Thus

$$t = t_0\gamma\,,$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,,$$

is the so-called gamma factor used in relativistic physics. Constant c denotes the speed of light, and v is the speed of a particle with respect to the observer (who is at rest).

Hence, particles decay at slower rate than if at rest. We express the decay law

$$N_2 = N_1 \mathrm{e}^{-\frac{\tau}{t}} = N_1 \mathrm{e}^{-\frac{h}{vt_0\gamma}}\,,$$

where $\tau = \frac{h}{v}$ represents the time that particle needs to travel from the balloon altitude to reach the ground; thus, the half-life is greater due to multiplication by γ-factor.

By combining these equations, we get

$$-\ln\frac{N_2}{N_1} = \frac{h}{vt_0\gamma}\,.$$

Now we simplify it to express $\frac{v}{c}$ ratio as

$$\frac{v}{c} = \frac{1}{\sqrt{1+\left(\frac{ct_0}{h}\right)^2 \ln^2\frac{N_1}{N_2}}} \doteq 0.94\,.$$

Hence, the particles traveled at 0.94 times the speed of light.

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Problem DG ... marble is falling off the window

A marble (a small hard ball) with a mass 50 g *charged itself by a charge* 50 mC*, and jumped to a narrow vertical tunnel with a height* $h = 50\,\mathrm{dm}$*. At the bottom end of the tunnel, there is a horizontally attached circle (the tunnel and the circle are concentric) with the radius* 50 cm *and the linear charge density* λ*. What is the minimum value of* λ*, if the marble wishes not to get lower than to height* $\frac{h}{5}$ *above the circle?*

Jarda was in the cinema to watch a new Bond movie.

First of all, we compute the electric potential induced by the charged circle. Let us denote the distance between the marble and the center of the circle as x. Then, all points on the circle with the radius R are equidistant from the marble, at the distance $\sqrt{R^2 + x^2}$. Therefore, the electrostatic potential energy of the marble with the charge Q is

$$E_\mathrm{e} = \frac{2\pi\lambda RQ}{4\pi\varepsilon_0\sqrt{R^2 + x^2}}\,.$$

If charges Q and λ are of the same polarity, the force acts upwards and is increasing in x. The first derivative of the potential is negative, but a force is defined as a negative gradient of potential energy; thus, it is positive in this case. The potential energy of the marble due to gravity is simply

$$E_\mathrm{p} = mgx\,,$$

where m is the mass of the marble. At point $x = h$, it's kinetic energy is zero. According to the problem statement, we seek another such point, and we denote its distance from the center as $x_0 = h/5$. The total energy is conserved, so

$$E_\mathrm{e} + E_\mathrm{p} + E_\mathrm{k} = \frac{\lambda RQ}{2\varepsilon_0\sqrt{R^2 + h^2}} + mgh = \frac{\lambda RQ}{2\varepsilon_0\sqrt{R^2 + x_0^2}} + mgx_0\,.$$

We express λ as

$$\lambda = \frac{2\varepsilon_0 mg\,(h - x_0)}{RQ\left(\frac{1}{\sqrt{R^2+x_0^2}} - \frac{1}{\sqrt{R^2+h^2}}\right)} = \frac{8\varepsilon_0 mgh}{5RQ\left(\frac{1}{\sqrt{R^2+\left(\frac{h}{5}\right)^2}} - \frac{1}{\sqrt{R^2+h^2}}\right)} = 2.0\cdot 10^{-9}\,\mathrm{C{\cdot}m^{-1}}\,.$$

If Q and λ were of the opposite polarity, there would be an attractive force between the marble and the circumference. If the marble were above the circle, it would be attracted downwards by both gravity and the electric force. Therefore, it would be impossible for the marble to stop above the circumference (however, the marble could stop below it). Indeed, both the charge-related quantities must be of the same polarity. Consequently, we look for positive λ.

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Problem DH ... attack on the train

One rascal decided to attack a moving train. He stepped in front of it on the rail and threw a bouncy ball against it with horizontal velocity $u = 13\,\mathrm{m{\cdot}s^{-1}}$*. At that time, the train was at a distance* $d = 17\,\mathrm{m}$ *away from him and was approaching at a steady speed. The bouncy ball bounced off the train elastically and reached the rascal after a time* $T = 1.5\,\mathrm{s}$ *from being* *thrown. The surprised rascal understood that the attack was unsuccessful, and it was time to disappear. What time does the rascal have to escape before a train passes through his current position?* *Jarda would like the trains to Prague to go faster.*

Let us denote t_1 the time from throwing the bouncy ball until its collision with the train and t_2 the time from the collision until the return of the bouncy ball to the rascal. Then

$$T = t_1 + t_2\,.$$

In time t_1 the bouncy ball and the train travel together a distance d, i.e.

$$d = ut_1 + vt_1\,,$$

where v is the speed of the train. In a system moving at the speed v together with the train, the train is at rest and during the collision, the bouncy ball collides with it at the speed $u + v$. Since the collision is perfectly elastic, the bouncy ball bounces off of it with the same speed. By switching back to the system connected to the ground, we get the speed of the bouncy ball after the bounce as the sum of the speed of the system connected to the train and the speed of the bouncy ball in this system, i.e.

$$u_1 = v + u + v = u + 2v\,.$$

With this speed the bouncy ball flies the time t_2 and between the collision and the rascal is a distance $d - vt_1$. So we have a third equation

$$d - vt_1 = (u + 2v)\,t_2\,.$$

We modify this equation to the form

$$d = (u + v)\,t_2 + v\,(t_1 + t_2)\,,$$

to which we substitute from the other two equations

$$d = d\frac{t_2}{t_1} + vT\,.$$

Substituting d at the third equation, we find the ratio of t_1 and t_2, which we insert into the previous expression. We get

$$d = d\frac{u}{u + 2v} + vT \quad \Rightarrow \quad v = \frac{d}{T} - \frac{u}{2}\,.$$

The time the problem statement asks for is

$$t = \frac{d - vT}{v} = \frac{T^2 u}{2d - Tu} = 2.0\,\mathrm{s}\,.$$

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Problem EA ... broken rheostat

Vojta had two light-emitting diodes (LEDs) – a red and a green one. He discovered that by sight, the red diode emits light when the current flowing through the diode is at least $I_1 = 10\,\mathrm{mA}$. Similarly, the green one needs current greater than $I_2 = 20\,\mathrm{mA}$. He was plugging these LEDs in series with a linear rheostat of resistance ranging from $R_1 = 100\,\Omega$ to $R_2 = 400\,\Omega$ into a circuit with source of voltage $U = 5\,\mathrm{V}$. However, Vojta accidentally broke the slider of the rheostat in a random position unknown to him. He thus tried to plug only the red LED and found it emitting light. What is the probability that the green LED will also light up when put instead of the red one? Both LEDs have the same operating voltage $U_\mathrm{p} = 1.7\,\mathrm{V}$.

Vojta played with Christmas lights.

Firstly, we need to determine the current flowing through the LED for a general value of rheostat's resistance R. Apparently

$$U = U_\mathrm{p} + IR \quad \Rightarrow \quad I = \frac{U - U_\mathrm{p}}{R}\,.$$

When calculating conditional probability, we use formula

$$P(I \geq I_2 | I \geq I_1) = \frac{P(I \geq I_2)}{P(I \geq I_1)}\,,$$

that expresses the sought probability in terms of probabilities that the current I is greater than the individual known currents. Let us now modify the expression of interest $P(I \geq I_i)$

$$P(I \geq I_i) = P\left(\frac{U - U_\mathrm{p}}{R} \geq I_i\right) = P\left(R \leq \frac{U - U_\mathrm{p}}{I_i}\right)\,.$$

The probability that the resistance R is greater than value $\frac{U-U_\mathrm{p}}{I_i}$ can be easily determined as

$$P\left(R \leq \frac{U - U_\mathrm{p}}{I_i}\right) = \frac{\frac{U-U_\mathrm{p}}{I_i} - R_1}{R_2 - R_1} = \frac{U - U_\mathrm{p} - R_1 I_i}{(R_2 - R_1)\, I_i}\,.$$

Putting everything together, we get

$$P(I \geq I_2 | I \geq I_1) = \frac{\frac{U-U_\mathrm{p}-R_1 I_2}{(R_2-R_1)I_2}}{\frac{U-U_\mathrm{p}-R_1 I_1}{(R_2-R_1)I_1}} = \frac{I_1}{I_2}\left(1 + R_1 \frac{I_1 - I_2}{U - U_\mathrm{p} - I_1 R_1}\right) \doteq 0.283\,.$$

We have also found that the general solution does not depend on R_2, which makes sense since the information about maximal resistance is already contained within the fact that we have managed to light up the red LED.

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Problem EB ... blured spectrum of rotations

We can never observe such sharp spectral lines in the star's spectrum as if we were to create the same light in the laboratory. One of the reasons is the rotational motion of the stars. To what width (expressed as the difference of the extreme wavelength $\Delta\lambda$) does this effect broaden the spectral line with the original frequency f_0? We assume that the other parameters are constant and that we could initially consider the line to be sharp. We also consider a star to be a sphere with radius R, which rotates at the equator with an angular velocity ω.

Karel likes astrophysics.

As we are only interested in the width of the spectral line, we don't care how fast a given star moves away from or approaches us. The fact that this value is not given or is to be considered constant is thus not essential.

One edge of the rotating star will move away from us at a speed of $R\omega$, while the other will approach us at the same speed. The wavelengths λ_1 and λ_2 of light incident from these ends can be determined using formula for Doppler effect

$$\lambda_1 = \lambda_0 \frac{c + R\omega}{c}\,,$$

$$\lambda_2 = \lambda_0 \frac{c - R\omega}{c}\,,$$

where λ_0 is the original wavelength of the spectral line and c is the speed of light in a vacuum. Thus, the spectral line is broadened to a width

$$\Delta\lambda = \lambda_1 - \lambda_2 = \frac{2R\omega\lambda_0}{c}\,.$$

We know the frequency f_0, for which $f_0 = c/\lambda_0$ applies. An equation for spectral line width can thus be recast as

$$\Delta\lambda = \frac{2R\omega\lambda_0}{c} = \frac{2R\omega}{f_0}\,.$$

This yields the desired result. In fact, the edges of the spectral line will be relatively dim compared to the line's center. At the same time, many other effects are acting both along the way and during the detection, which causes spectral lines to broaden. However, this formula can be used in practice to approximate the speed of rotation.

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Problem EC ... watch your head

Jarda hates taking out the garbage at the dormitory, so he usually throws it out of the window. One day, he threw out two balls, both with radius $R = 8\,\mathrm{cm}$. The first was of mass $m_1 = 85\,\mathrm{g}$ and the other was of mass $m_2 = 123\,\mathrm{g}$. How many times will the kinetic energy of the second ball be greater than that of the first one upon impact? Jarda lives on the 16th floor.

Jarda was wondering whether such a ball could be a murderous tool.

Let us denote the quantities from the task (in the same order) as R, m_1 a m_2. Because the balls are falling from an enormous height, we expect their speed to be stable, and the resistance force and gravity to be in equilibrium. Since the flow obviously will not be laminar, we can write this equality as

$$kv_i^2 = F_\mathrm{r} = F_g = m_i g$$

where $i = 1, 2$. We adjust the equation

$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} \frac{g}{k} m_i^2 \,.$$

Since g and k are the same for both of the balls, the ratio of balls' kinetic energies is equal to the second power of their masses' ratio, thus

$$\frac{E_{\mathrm{c}2}}{E_{\mathrm{c}1}} = \left(\frac{m_2}{m_1}\right)^2 \doteq 2.09 \,.$$

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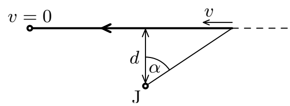
Problem ED ... duck



Jarda sits on the shore of a pond with calm surface. Suddenly a duck begins to land on the surface of the pond in a specific way. It moves along a line, which is at distance $d = 2.8\,\mathrm{m}$ from Jarda. After touching the water, the duck slows down uniformly, and her motion creates waves on the surface of the pond. The waves reach Jarda at angle $\alpha = 7.0^\circ$ (see picture), and in time $t_1 = 10.3\,\mathrm{s}$ (since it touched the water). The duck stops moving in time $t_2 = 3.1\,\mathrm{s}$. What distance on a water surface did the duck travel? The speed of waves on the surface is $c = 0.30\,\mathrm{m{\cdot}s^{-1}}$. *Jarda feeds the birds in winter.*

Firstly, suppose that the duck landed (touched the water) on the aforementioned line at a distance $\frac{d}{\cos\alpha} = 2.82\,\mathrm{m}$ from Jarda and that waves propagate by speed c directly towards him. They reach him at time t_1 after landing, and thus they traveled distance $ct_1 = 3.09\,\mathrm{m}$. However, this is not equal to the value we determined earlier.

This means that the duck must have had greater speed than c during the slowing down for some time. Thus, the waves propagate from a different place than where the duck landed. We denote the speed of the duck as v. If $v > c$, then waves are triangle-shaped with vertex angle $\beta = \arcsin\frac{c}{v}$. These waves propagate perpendicular to their wavefront. From the figure, it is clear that $\beta = \alpha$. Therefore, we know the duck's speed at the moment when the waves (that later reached Jarda at angle α) started to propagate from the duck. This speed is

$$v = \frac{c}{\sin\alpha}$$

and waves were formed at

$$t_\mathrm{v} = \frac{d}{\cos\alpha c} \,,$$

prior to reaching Jarda. If we denote time when the duck landed on water as $t_0 = 0\,\mathrm{s}$, the waves were formed at time

$$t_1 - t_\mathrm{v} = t_1 - \frac{d}{\cos\alpha c}$$

when the duck had speed $v = \frac{c}{\sin\alpha}$. The duck has zero speed at time t_2, which means that it decelarated by v in time $t_2 - (t_1 - t_\mathrm{v})$. Hence, its acceleration is

$$a = \frac{v}{t_2 - (t_1 - t_\mathrm{v})} = \frac{\frac{c}{\sin\alpha}}{t_2 - t_1 + \frac{d}{\cos\alpha c}}\,.$$

Therefore, the distance that the duck traveled on the water is

$$s = \frac{1}{2}at_2^2 = \frac{ct_2^2}{2\left((t_2 - t_1)\sin\alpha + \tan\alpha\frac{d}{c}\right)} = 5.4\,\mathrm{m}\,.$$

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Problem EE ... we're setting ants on fire

On a beautiful sunny day, at the noon of the summer solstice in Prague, when the air temperature is 35 °C *we take a convex lens with a diameter* $D = 6\,\mathrm{cm}$, *and we place it perpendicularly to the sun rays. We put a little black ball with high thermal conductivity and radius* $r = 1\mathrm{mm}$ *to the focal point of the magnifying glass behind the lens.*

What will be the equilibrium temperature of the black ball? Heat transfer by convection can be neglected. Only 60 % *of the solar radiation passes through the atmosphere, and* 65 % *of the radiation energy passes through the glass lens.*

Jarda wanted to magnify insect, but it turned out badly.

The solar radiation enters the Earth's atmosphere with intensity corresponding to the solar constant $K = 1\,361\,\mathrm{W{\cdot}m^{-2}}$. Only $K_\mathrm{Z} = 0.6K \doteq 817\,\mathrm{W{\cdot}m^{-2}}$ passes through the atmosphere to the Earth's surface. This value is reduced to $K_\mathrm{C} = 0.65K_\mathrm{Z} \doteq 531\,\mathrm{W{\cdot}m^{-2}}$ after passing through the lens. Since the lens is oriented perpendicularly to the rays, the total power passing through the magnifying glass is

$$P = 0.6 \cdot 0.65KS = \frac{0.39\pi D^2 K}{4}\,.$$

The lens directs this power to the focal point, where the little black ball is placed.

In the equilibrium state, this little ball receives the same amount of energy that it radiates. It receives the energy from the Sun through the magnifying glass and also from its surroundings due to the radiation. The equation of equilibrium is

$$4\pi r^2\sigma T^4 = 4\pi r^2\sigma T_\mathrm{o}^4 + \frac{0.39\pi D^2 K}{4}\,,$$

where T is the temperature of the little ball, σ is the Stefan–Boltzmann constant, and $T_\mathrm{o} = 308.15\,\mathrm{K}$ is the temperature of the surroundings. We get

$$T = \sqrt[4]{T_o^4 + \frac{0.39D^2K}{16r^2\sigma}} \doteq 1\,206\,\mathrm{K} = 933\,\mathrm{^\circ C}\,.$$

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Problem EF ... charged helicopter

FYKOS organizers decided to go on a helicopter trip, and now they are in the air, above the north magnetic pole, where they just measured the magnitude of a magnetic field $B = 65\,\mu\mathrm{T}$. The length of the rotor blade from the rotor to the end of the blade is $L = 6\,\mathrm{m}$. What voltage arises between the end of the rotor blade and the helicopter main rotor, which rotates with frequency $f = 3\,\mathrm{s}^{-1}$? *Jarda casually takes helicopter rides to the North Pole.*

Let us assume a small rotor blade element of a length $\mathrm{d}r$ and at a distance r from the center of rotation. A magnetic force in the direction along the blade and of magnitude $F_\mathrm{m} = Bvq$ acts on the electrons of this element. In equilibrium, it is compensated by an induced electric force of magnitude $F_\mathrm{e} = Eq$. Speed v depends on the distance from the axis of rotation as $v = \omega r$. From the equality of these forces, we get

$$E = B\omega r\,.$$

We obtain the voltage by integration of electric intensity along the blade

$$U = \int_0^L B\omega r\,\mathrm{d}r = \frac{1}{2}B\omega L^2\,.$$

Therefore, the voltage between the helicopter main rotor and the end of the blade is

$$U = \frac{1}{2}B\omega L^2 = \pi B f L^2 = 22\,\mathrm{mV}\,.$$

We can see that the voltage is negligible.

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Problem EG ... long wait

Lego thinks he waits for the metro pretty often for too long (sometimes even 10 minutes!). So he calculated what percentage of the total waiting time is made up by waits longer than half of the interval (time between the departure of previous and arrival of the following metro, which Lego is waiting for). What result did Lego obtain if he calculated it correctly? By the way, Lego always comes to the metro station at a random moment in the interval. *Lego hates waiting.*

Let us denote the time period between departure of one and arrival of another metro as T. Then the Lego's waiting time is from interval $(0, T)$, and from the problem, we know, that all times from this interval are of equal probability. Precisely, the probability that Lego's waiting time is from interval $(t, t + \mathrm{d}t)$ is for all times $t < T$ equal to $\mathrm{d}t/T$. Average wait will, therefore, last

$$\bar{t} = \int_0^T t\frac{\mathrm{d}t}{T} = \frac{1}{T}\left[\frac{t^2}{2}\right]_0^T = \frac{T}{2}\,.$$

If we think about it, this results seems pretty intuitive. However, what portion of this time is made up by waits longer than $T/2$? We determine it by similar computation

$$\bar{t}_{t>T/2} = \int_{T/2}^T t\frac{\mathrm{d}t}{T} = \frac{1}{T}\left[\frac{t^2}{2}\right]_{T/2}^T = \frac{1}{T}\left(\frac{T^2}{2} - \frac{T^2}{8}\right) = \frac{3}{8}T\,.$$

Hence, despite making only half of the number of all waits, waits longer than one half of the time interval make most of the total waiting time

$$\frac{\frac{3}{8}T}{\frac{1}{2}T} = \frac{3}{4} = 75\,\%\,.$$

This is probably the reason why it seems to Lego that he is always waiting for too long.

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Problem EH ... terrain advantage

Imagine you are standing in the middle of a large inclined plane tilted at angle $\alpha = 25^\circ$. What is the ratio of maximal ranges of the projectile shot directly uphill to projectile shot the opposite way? Assume the maximal projectile velocity does not depend on the shooting angle.

Dodo was trying to shoot uphill.

Let us fix the maximum projectile velocity v. If we shot any slower, the range would be lower. This problem can be solved in different ways. The most straightforward is calculating the range for a given angle of inclination as a function of projectile angle and further maximizing this function by differentiating with respect to the projectile angle. However, this procedure is relatively time-consuming, so in our solution, we will use the fact that all places that can be hit lie below the safety parabola. We set the origin of the coordinate system at the point of firing and the x-axis points in the horizontal plane in uphill orientation. We can describe this in a two-dimensional section with the greatest inclination as

$$y = x \tan\alpha\,.$$

We obtain the equation of safety parabola $y = \alpha x^2 + \beta$ using the position of its vertex at a point at a height $h = \frac{v^2}{2g} = \beta$ above the point of firing (can be found from law of conservation of energy) and the maximum range in the horizontal plane $d = \frac{v^2}{g} = \sqrt{\frac{\beta}{-\alpha}}$ (obtained from vertical and horizontal trajectories). The resulting parabola's equation is

$$y = -\frac{g}{2v^2}x^2 + \frac{v^2}{2g}\,.$$

To find the position of the points of impact, let us set the equality for y-coordinates of both curves, which provides us with a quadratic equation

$$\frac{g}{2v^2}x^2 + x\tan\alpha - \frac{v^2}{2g} = 0\,,$$

whose two roots are the sought ranges measured horizontally. This is suffiecient as we seek only ratio of them so cosines cancel out as the slope is constant

$$x_{+,-} = \frac{v^2}{g}\left(-\tan\alpha \pm \sqrt{\tan^2\alpha + 1}\right) = \frac{v^2}{g}\frac{-\sin\alpha \pm 1}{\cos\alpha}\,.$$

Sign of the result only distinguishes the direction. Hence, we easily obtain the ratio as

$$w = \left|\frac{x_+}{x_-}\right| = \frac{1-\sin\alpha}{1+\sin\alpha} \doteq 0.406\,.$$

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Problem FA ... cube in slip

An ice cube of mass $m_0 = 7.34\,\mathrm{g}$, and temperature $T = 0\,{}^\circ\mathrm{C}$ is placed on a long, hot inclined plane tilted at angle $\alpha = 15^\circ$. Since that moment, the cube absorbs heat at rate $P = 1\,\mathrm{kW}$. Determine the work done by the gravitational force on ice acceleration. Assume that the ice cube melts uniformly and only from the bottom base. Neglect all resistance forces.

Vojta was watching old commercials.

While moving, cube is subject to a force

$$F(t) = m(t)\,g\sin\alpha\,.$$

We determine work done by gravitational force as

$$W = \int_0^{s_1} m(t)\,g\sin\alpha\,\mathrm{d}s = \int_0^{t_1} m(t)\,v(t)\,g\sin\alpha\,\mathrm{d}t\,.$$

At first, we need to know the total time the ice moves. Let l_t be the specific latent heat of fusion of ice. Let us find the time in which the entire ice cube melts

$$t_1 = \frac{Q}{P} = \frac{m_0 l_\mathrm{t}}{P}\,.$$

Furthermore, we need to express the speed of the ice cube as a function of time. Note that the acceleration remains constant, with magnitude

$$a = g\sin\alpha\,.$$

Thus, it satisfies

$$v = tg\sin\alpha\,.$$

Lastly, we need to express the mass of the ice as a function of time. Note that in time t, the ice cube receives heat Pt and consequently, ice of mass Pt/l_t melts. Now, we have everything prepared and can write integral as

$$W = \int_0^{t_1} \left(m_0 - \frac{Pt}{l_\mathrm{t}}\right) tg^2\sin^2\alpha\,\mathrm{d}t = g^2\sin^2\alpha\int_0^{t_1} m_0 t - \frac{Pt^2}{l_\mathrm{t}}\,\mathrm{d}t = g^2\sin^2\alpha t_1^2\left(\frac{m_0}{2} - \frac{Pt_1}{3l_\mathrm{t}}\right)\,,$$

where after substitution for t_1 we get

$$W = \frac{g^2\sin^2\alpha m_0^3 l_\mathrm{t}^2}{6P^2} \doteq 0.047\,\mathrm{J}\,.$$

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Problem FB ... double infinity of lenses

Jarda got a very large number of really thin convex lenses for Christmas. He placed them very closely one behind the other so that the first one had a focal length f, the second one $2f$, a third $4f$ and so on. He also found a concave mirror with radius of curvature f, which he placed right behind this row of lenses. He inserted a luminous object into the focus of the first lens. What distance in front of the mirror did its image appear? If it appeared behind the mirror, give a negative answer. *Jarda is bored with problems about infinite circuits, this is a novelty!*

First, let's look at how the beam passes through two lenses placed one behind the other. Let a be the distance of the object in front of the first lens with focal length f_1. Just behind it is a second lens with focal length f_2. Using the Gaussian lens equation, we find the position of the image a' of the object as

$$a' = \frac{af_1}{a - f_1}\,.$$

This distance is positive if the object is imaged behind the first lens. The image will be viewed through the second lens, with the pattern distance now equal to $-a'$. The second lens displays the object as

$$a'' = \frac{-a'f_2}{-a' - f_2} = \frac{-\left(\frac{af_1}{a-f_1}\right)f_2}{-\frac{af_1}{a-f_1} - f_2} = \frac{af_1f_2}{af_1 + af_2 - f_1f_2} = \frac{aF}{a - F}\,,$$

where $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$. So we derived an interesting relation that for two lenses in close vicinity to each other, their total optical power is the sum of the optical powers of the two lenses.

The relationship can be iterated and it is evident that the total optical power of all Jarda's lenses in a row is

$$\frac{1}{F} = \frac{1}{f}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) = \frac{2}{f}\,,$$

From here $F = \frac{f}{2}$. The focal length of this lens series is thus half that of the first lens. We can calculate the infinite sum above for example from the relation for an infinite geometric series. The set of these lenses will therefore image an object located at a distance f in front of it to a distance

$$x = \frac{f\frac{f}{2}}{f - \frac{f}{2}} = f\,.$$

Now let's move on to the image on the concave mirror. It has a focal length equal to one half of the radius of curvature. For a point with distance $-f$ (the negative sign is due to the fact that the lens system imaged the object behind the mirror) the image will be in distance

$$x' = \frac{-x\frac{f}{2}}{-x - \frac{f}{2}} = \frac{f}{3}\,.$$

Now the object is shown at a distance $x' = \frac{f}{3}$ in front of the mirror.

Further imaging is again performed using a system of lenses. The object is now located $x' = -\frac{f}{3}$ behind the lens system. For the last time we use the imaging equation

$$x'' = \frac{x'\frac{f}{2}}{x' - \frac{f}{2}} = \frac{\frac{-f}{6}}{-\frac{1}{3} - \frac{1}{2}} = \frac{f}{5}\,.$$

The object was imaged through the entire system to a distance $\frac{f}{5}$ in front of both the mirror and the lenses.

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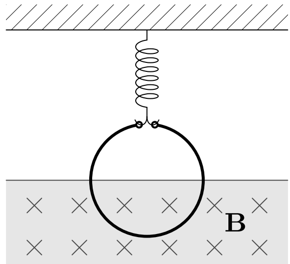
Problem FC ... oscillatory voltage

Metal hoop with radius r and mass m is suspended on a spring with stiffness k in such a way, that its lower half is placed in a magnetic field with induction B, which is perpendicular to the hoop, as shown in the figure. What will be the peak voltage on the hoop, if we let it oscillate with the displacement amplitude equal to r? Assume that the voltage is measured at the hinge point in a place where the hoop is disconnected.

Vojta was inventing innovative voltage sources.



Notice that no current will be passing through the hoop, since it is not connected. That means we don't have to deal with damping. From Faraday's law of induction, we can write

$$U = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -B\frac{\mathrm{d}S}{\mathrm{d}t} = -Bl\frac{\mathrm{d}y}{\mathrm{d}t} = -Blv$$

where y is the displacement of the spring, l is the distance between two points on a hoop lying on the border of the magnetic field, and v is the instantaneous velocity of the hoop. Let us now express the displacement as a function of time

$$y = r\cos\omega t\,,$$

from where we can directly determine the instantaneous velocity as

$$v = \frac{\mathrm{d}y}{\mathrm{d}t} = -\omega r\sin\omega t\,.$$

Using the Pythagorean theorem, we can also notice

$$l = 2\sqrt{r^2 - y^2} = 2r\sqrt{1 - \cos^2\omega t} = 2r\left|\sin\omega t\right|\,.$$

Substituting these expressions into expression for voltage we get

$$U = 2Br^2\omega\left(\sin\omega t\left|\sin\omega t\right|\right)\,,$$

where the expression in parentheses ranges from -1 to 1. Expressing the spring angular frequency in terms of given quantities, we can express the amplitude of voltage as

$$U_\mathrm{A} = 2Br^2\sqrt{\frac{k}{m}}\,.$$

We can also solve the problem by directly expressing the area of a circular segment as a function of time, which yields the same answer. To complement the solution, the graph depicting the voltage as a function of time is included.

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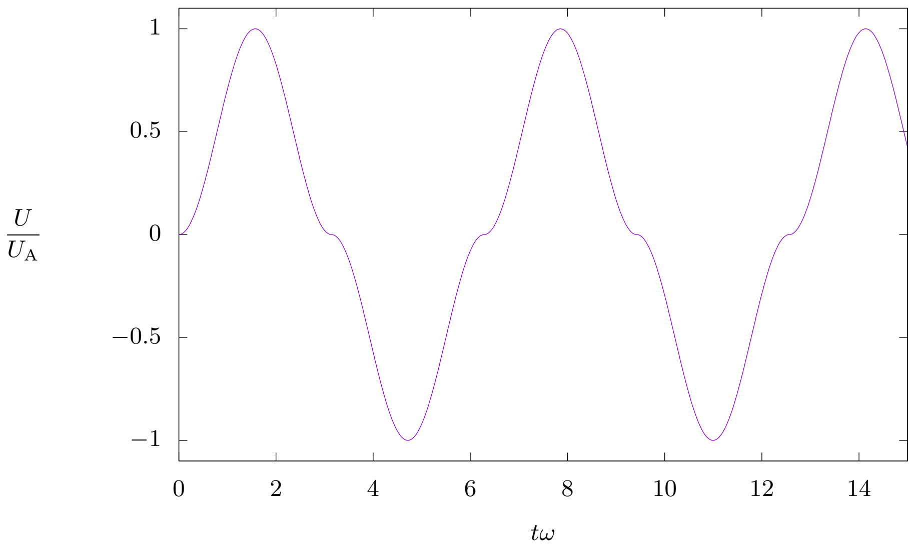

Fig. 1: Voltage as a function of time.

Problem FD ... exhausted Sun

Due to thermonuclear fusion, the Sun loses its mass, which transforms into radiation. Determine the time in which the distance between the Sun and the Earth changes by 1 m *because of this phenomenon. Assume the Earth moves in a circular orbit. Provide answer in years.*

Jarda noticed that the world was not the same anymore.

The Earth moves on a circular orbit thanks to the centripetal force induced by Sun's gravity. We can express this as

$$m\frac{v^2}{r} = \frac{MmG}{r^2}\,,$$

where m is the Earth's mass, M is the mass of the Sun, v is the velocity of Earth's movement around the Sun, and r is the distance between the Earth and the Sun. Now we multiply this equation by r^2 and differentiate it with respect to time

$$2vr\frac{\mathrm{d}v}{\mathrm{d}t} + v^2\frac{\mathrm{d}r}{\mathrm{d}t} = G\frac{\mathrm{d}M}{\mathrm{d}t}\,.$$

In the system, the law of conservation of angular momentum per (Earth's) unit mass holds at each time; we can write it as $l = rv = \mathrm{const}$. Furthermore, we differentiate this equality with respect to time, which provides us with the following formula

$$v\frac{\mathrm{d}r}{\mathrm{d}t} + r\frac{\mathrm{d}v}{\mathrm{d}t} = 0\,.$$

We substitute these expressions into the differentiated equation above and get

$$2vr\left(-\frac{v}{r}\frac{\mathrm{d}r}{\mathrm{d}t}\right)+v^2\frac{\mathrm{d}r}{\mathrm{d}t}=-v^2\frac{\mathrm{d}r}{\mathrm{d}t}=G\frac{\mathrm{d}M}{\mathrm{d}t}\,.$$

The change of the Sun's mass in time is determined by the transformation of its rest mass into radiation. We can write this as

$$\frac{\mathrm{d}M}{\mathrm{d}t}=-\frac{L_\odot}{c^2}\,,$$

where $L_\odot$ is nominal solar luminosity. If we substitute and realize that changes are of small order, we can proceed from differentials to finite changes, obtaining

$$\Delta t=v^2c^2\frac{\Delta r}{GL_\odot}\,.$$

By expressing the square of velocity from the first equation, and number substitution, we get

$$\Delta t=\frac{M_\odot c^2}{L_\odot}\frac{\Delta r}{r}\doteq 99\,\mathrm{years}\,.$$

Due to the change in Sun's mass via radiation, the Earth moves away from it by one meter in almost a hundred years. This effect is truly negligible yet measurable by radar.

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Problem FE ... marble didn't guess

In the previous two problems, the marble (a small hard ball) always stopped somewhere during its journey. Now, the marble wants to float in the air. The marble turned on a monochromatic light source at the bottom, perpendicular to the ground, pointing upwards with intensity P. Then the marble put on a perfectly reflective coat. Afterward, the marble jumps off the window at a height H, directly into the light beam. The marble has a radius R and a mass m. Unfortunately, the marble did not compute the needed intensity correctly, and the light beam cannot keep it floating in the air. At what speed does the marble hit the ground?

Jarda will not try this at the dormitory either.

Firstly, we compute the force by which the light beam acts on the marble. Consider a photon with momentum $p_1=\frac{hf}{c}$ moving from the source directly upwards, which hits the marble. We parametrize the point of interaction by an angle φ formed by the line connecting the marble's center and the point of impact on the cylindrical surface, with the vertical line passing through the marble's center. Satisfying Snell's law, the photon bounces such that its momentum points $\pi-2\varphi$ away from the vertical axis.

Thus, the vertical component of the photon's momentum is $p_2=p_1\cos(\pi-2\varphi)$. The change in the photon's vertical momentum is therefore,

$$\Delta p=p_1-p_2=p_1\left(1-\cos\left(\pi-2\varphi\right)\right)=p_1\left(1+\cos 2\varphi\right)\,.$$

The total change of marble's horizontal momentum is zero due to the symmetry, so we will not consider it further.

We know the energy of photon satisfies $E = hf$. Therefore, the number of photons emitted per area unit per time unit is

$$n = \frac{P}{hf}\,.$$

Hence, the change in momentum per area unit (actually pressure) is

$$n\Delta p = \frac{P}{c}\left(1 + \cos\left(2\varphi\right)\right)\,.$$

We determine the total force caused by the pressure of the radiation by integrating this change in the momentum per area unit with respect to the area. Regarding the φ, we integrate over the circle element. Note that using simple geometry, we find that it satisfies

$$\mathrm{d}S = 2\pi R\sin\varphi R\,\mathrm{d}\varphi\cos\varphi = \pi R^2\sin 2\varphi\,\mathrm{d}\varphi\,.$$

The magnitude of the force is

$$F_z = \int_0^{\frac{\pi}{2}} \frac{P}{c}\left(1 + \cos 2\varphi\right)\pi R^2\sin 2\varphi\,\mathrm{d}\varphi \;=\frac{\pi R^2 P}{c}\int_0^{\frac{\pi}{2}}\left(1 + \cos 2\varphi\right)\sin 2\varphi\,\mathrm{d}\varphi\,.$$

We simplify the integrand to

$$\sin 2\varphi + \frac{\sin 4\varphi}{2}\,,$$

which provides us

$$F_z = -\frac{\pi R^2 P}{c}\left[\frac{\cos 2\varphi}{2} + \frac{\cos 4\varphi}{8}\right]_0^{\frac{\pi}{2}} = \frac{\pi R^2 P}{c}\,.$$

Thus, the total force acting on the marble is

$$F = mg - F_z = mg - \frac{\pi R^2 P}{c}$$

and from the law of conservation of energy, we can determine the sought velocity as

$$v = \sqrt{\frac{2FH}{m}} = \sqrt{2H\left(g - \frac{\pi R^2 P}{mc}\right)}\,.$$

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Problem FF ... zigzag

Rickon Stark is moving away from Ramsay Bolton at constant speed $v_r = 3.6\,\mathrm{m{\cdot}s^{-1}}$. Let us imagine that he chooses a better strategy this time than in the source material. Instead of running in a straight line, he will now run zigzag. On top of running forward, he will run at speed $v_\varphi = 2.0\,\mathrm{m{\cdot}s^{-1}}$ in direction perpendicular to the direction of v_r. Ramsay is standing in the center of a circle sector of angle $\alpha = 20^\circ$. Rickon is running from one edge of the sector to the other (and back) all the time. In the beginning, he is located $r_0 = 8\,\mathrm{m}$ from Ramsay at the edge of the sector. Ramsay knows that after he shoots, the arrow will land at a distance

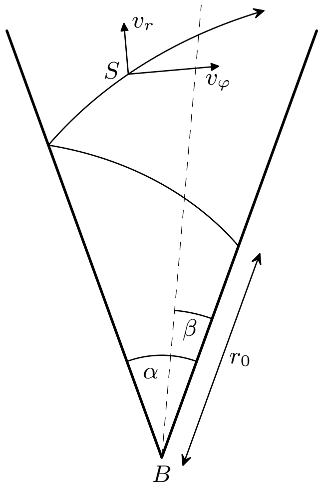

Fig. 2: The circular section in which Rickon moves.

$r_{\mathrm{R}} = 65\,\mathrm{m}$ *from him. At what angle in the segment does he have to aim in order to hit this time as well?* *Jáchym was a Ramsay fan but wanted the story to make sense.*

Let r denote the distance of Rickon from Ramsay. From the problem statement it is apparent, that the distance r as a function of time is given by $r = r_0 + v_r t$. In any given moment Rickon has angular velocity ω, for which $\omega r = v_\varphi$ holds. This means, that angular velocity is not constant (it is actually decreasing). If we want to find out the total angle traveled by Rickon at given time, we need to use integration

$$\varphi = \int_0^t \omega \,\mathrm{d}\tau = \int_0^t \frac{v_\varphi}{r_0 + v_r \tau} \,\mathrm{d}\tau = \left[\frac{v_\varphi}{v_r} \ln\left(r_0 + v_r \tau\right)\right]_0^t = \frac{v_\varphi}{v_r} \ln\left(1 + \frac{v_r t}{r_0}\right) = \frac{v_\varphi}{v_r} \ln\left(\frac{r}{r_0}\right) .$$

If we look at φ as a function of distance r, the total angle traveled by Rickon, before the arrow hits him, is given by

$$\varphi = \frac{v_\varphi}{v_r} \ln \frac{r_{\mathrm{R}}}{r_0} = 66.7^\circ .$$

Let's denote the angle under which Ramsay has to aim as β. We measure this angle from the edge the Rickon started at. To find out β we simply need to subtract the angle 2α from φ as many times as needed to get the angle $\beta' \in \langle 0, 2\alpha)$. If $\beta' \in \langle 0, \alpha)$, then $\beta = \beta'$, and we have our solution. If $\beta' \in \langle \alpha, 2\alpha)$ then, since we are measuring β from the edge Rickon started at, our solution is $\beta = 2\alpha - \beta'$. With our initial values we get $\varphi = 66.7^\circ$, $\beta' = 26.7^\circ$ and the solution is $\beta = 40.0^\circ - 26.6^\circ = 13.3^\circ$.

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Problem FG ... magnetic (almost)levitation

On a non-conductive horizontal plane, there lies a point of mass $m = 0.8\,\mathrm{kg}$ and charge $q = 3\,\mathrm{C}$. The coefficient of friction between the point and the plane is $f = 0.4$. At the same time, there is a constant homogeneous horizontal magnetic field $B = 0.5\,\mathrm{T}$. What is the maximum distance this point can travel if its initial horizontal velocity is $v_0 = 4\,\mathrm{m{\cdot}s^{-1}}$?

On his way to Prague, Jarda had a dream about fast trains.

In addition to the constant gravitational force, the magnetic force will also act on the point. This force is perpendicular to the magnetic field and the instantaneous velocity, both of which lie in the horizontal plane. Therefore, the magnetic force will act solely in the vertical direction – just like gravity.

In order to travel the greatest distance possible, the friction force between the point and the plane must be the smallest, i.e., the normal force must be minimal. This occurs when the magnetic field is perpendicular to the velocity vector and the magnetic force points upward. Then it satisfies equation

$$F_\mathrm{n} = mg - Bvq\,,$$

where v is the magnitude of the instantaneous velocity. We can see that point does not detach from the plane by substituting the numbers. Since the friction decelerates the point, we can write Newton's second law as

$$F = ma = -m\frac{\mathrm{d}v}{\mathrm{d}t} = f\left(mg - Bvq\right)\,.$$

This differential equation can be solved by separation of variables (the Fourier method)

$$m\frac{\mathrm{d}v}{Bvq - mg} = f\,\mathrm{d}t \quad \Rightarrow \quad \ln(Bvq - mg) = \frac{Bqf}{m}t + C_0\,,$$

which yields

$$v = C\mathrm{e}^{\frac{Bqf}{m}t} + \frac{mg}{Bq}\,.$$

The integration constant C can be determined using initial condition $v = v_0$ at $t = 0$. Thus,

$$v = \left(v_0 - \frac{mg}{Bq}\right)\mathrm{e}^{\frac{Bqf}{m}t} + \frac{mg}{Bq}\,.$$

The point stops when its velocity is zero, i.e., at a time

$$t_1 = \frac{m}{Bqf}\ln\frac{mg}{(mg - Bqv_0)}\,.$$

By integrating over time we get a covered distance as

$$s = \frac{m}{Bqf}\left(v_0 - \frac{mg}{Bq}\right)\mathrm{e}^{\frac{Bqf}{m}t} + \frac{mg}{Bq}t + C_2\,.$$

A constant C_2 can be also determined using intitial condition $s = s_0 = 0$ at $t = 0$, therefore

$$s = \frac{m}{Bqf}\left(v_0 - \frac{mg}{Bq}\right)\left(\mathrm{e}^{\frac{Bqf}{m}t} - 1\right) + \frac{mg}{Bq}t\,.$$

By substituing $t = t_1$, we obtain the maximum distance this point can travel

$$s = \frac{m^2 g}{B^2 q^2 f} \ln \frac{mg}{(mg - Bqv_0)} - \frac{mv_0}{Bqf} = 4.76\,\mathrm{m}\,.$$

To compare, without a magnetic field, such a point would travel just a little over 2 m.

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Problem FH ... thirteen shots

Consider three cubic water containers, the first of which has a base with an area of $S_a = 72.9\,\mathrm{cm^2}$, the second $S_b = 94.5\,\mathrm{cm^2}$ and the third $S = 3.1\,\mathrm{cm^2}$, while the symbols S_a, S_b and S are also used to label the containers themselves. Initially, the containers S and S_b are connected and the water in them reaches a height of $h_b = 8.1\,\mathrm{cm}$, while the level in the first container is $h_a = 19.3\,\mathrm{cm}$. Then we disconnect the containers S and S_b, join S and S_a, and wait until the levels equalize. Then, we disconnect them, reconnect S and S_b, and wait until the levels equalize again. We will perform the above-described process a total of thirteen times. What will be the final water level in the container S after the last iteration?

Jáchym had to continue the tradition of the famous "thirteen" tasks from previous years.

An important first step in solving this problem is to establish a suitable and clear labeling of all necessary variables. The level in the containers S_a and S_b after the i-th step will be denoted x_i and y_i respectively. The initial values are $x_0 = h_a$ and $y_0 = h_b$.

After we connect the containers S and S_a for the the first time, the levels will equalize to x_1, while initially it was y_0 and x_0 respectively. The conservation of volume implies

$$(S + S_a)\, x_1 = S y_0 + S_a x_0\,.$$

Then we connect the container S and S_b, in which the levels are x_1 and y_0. Once equalized, the level will be y_1, for which it holds

$$(S + S_b)\, y_1 = S x_1 + S_b y_0\,.$$

If we define $a = S/S_a$ and $b = S/S_b$, we can generalize these results as

$$x_{i+1} = \frac{x_i + a y_i}{1 + a}\,,$$
$$y_{i+1} = \frac{y_i + b x_{i+1}}{1 + b}\,,$$

where we substitute into the second equation from the first and obtain

$$y_{i+1} = \frac{b x_i + (1 + a + ab)\, y_i}{(1 + a)\,(1 + b)}\,.$$

For the sake of clarity, we redefine the constants as

$$x_{i+1} = \alpha x_i + \beta y_i\,,$$
$$y_{i+1} = \gamma x_i + \delta y_i\,.$$

The transformation relations thus read as follows

$$\alpha = \frac{1}{1+a}\,, \qquad \beta = \frac{a}{1+a}\,,$$
$$\gamma = \frac{b}{(1+a)(1+b)}\,, \quad \delta = \frac{1+a+ab}{(1+a)(1+b)}\,.$$

This system of equations can be solved in several ways. The fastest of them would be a transcription into the language of linear algebra as it holds

$$\mathbf{x}_{i+1} = A\mathbf{x}_i \quad \Rightarrow \quad \mathbf{x}_{13} = A^{13}\mathbf{x}_0\,,$$

where $\mathbf{x}_i = (x_i, y_i)$ and A is a matrix of coefficients α, β, γ, δ. Then we could find a diagonal form of a matrix A, in which it is straightforward to calculate its power and convert it back. Unfortunately, this procedure does not fall under high school mathematics, so we will present a more intuitive solution.

The goal is to separate the equations so that each contains only one variable. We achieve this by a suitable transformation. For the constant k let's define $z_i = x_i + ky_i$, then

$$z_{i+1} = x_{i+1} + ky_{i+1} = (\alpha + k\gamma)\,x_i + (\beta + k\delta)\,y_i = (\alpha + k\gamma)\left(x_i + \frac{\beta + k\delta}{\alpha + k\gamma} y_i\right)\,.$$

Since k is arbitrary, let's choose it such

$$k = \frac{\beta + k\delta}{\alpha + k\gamma} \quad \Rightarrow \quad \gamma k^2 + (\alpha - \delta)\,k - \beta = 0 \quad \Rightarrow \quad k_{1,2} = \frac{\delta - \alpha \pm \sqrt{(\delta - \alpha)^2 + 4\gamma\beta}}{2\gamma}\,.$$

For $K = \alpha + k\gamma$ it holds

$$z_{i+1} = (\alpha + k\gamma)\,(x_i + ky_i) = Kz_i\,,$$

which applies for both possible definitions of z (according to the possible values of the constant k). For clarity, we divide them into separate variables μ_i and ν_i, we get

$$\mu_i = x_i + k_1 y_i \quad \Rightarrow \quad \mu_{i+1} = K_1 \mu_i\,,$$
$$\nu_i = x_i + k_2 y_i \quad \Rightarrow \quad \nu_{i+1} = K_2 \nu_i\,.$$

Note that these new variables are not coupled – μ_{i+1} depends only on μ_i, not on ν_i. This makes it easy to calculate their value for any i as $\mu_i = K_1^i \mu_0$.

Now we need to transform back to the x and y variables. To do this, we simply express x_i and y_i from the definitions of μ_i and ν_i. Substituting $i = 13$ into the result gives us the water levels after thirteen repetitions

$$x_i = \frac{k_2\mu_i - k_1\nu_i}{k_2 - k_1} \quad \Rightarrow \quad x_{13} = \frac{k_2\mu_{13} - k_1\nu_{13}}{k_2 - k_1} = \frac{k_2 K_1^{13}\mu_0 - k_1 K_2^{13}\nu_0}{k_2 - k_1}\,,$$
$$y_i = \frac{\mu_i - \nu_i}{k_1 - k_2} \quad \Rightarrow \quad y_{13} = \frac{\mu_{13} - \nu_{13}}{k_1 - k_2} = \frac{K_1^{13}\mu_0 - K_2^{13}\nu_0}{k_1 - k_2}\,.$$

However, we are only interested in the water level in the container S, for which it holds $h = y_{13}$. All that remains is to use the values given in the problem statement, calculate all constants used and finally plug them into the relation

$$h = \frac{K_1^{13}\,(x_0 + k_1 y_0) - K_2^{13}\,(x_0 + k_2 y_0)}{k_1 - k_2} \doteq 11.1\,\mathrm{cm}\,.$$

It might seem that a general solution has to be lenghty. Surprisingly, this is not the case as

$$k_1 = \frac{a\,(1+b)}{b}\,, \qquad K_1 = 1\,,$$
$$k_2 = -1\,, \qquad K_2 = \frac{1}{(1+a)\,(1+b)}\,.$$

This leads to a relatively simple result

$$h = \frac{bh_a + a\,(1+b)\,h_b - b\,[(1+a)\,(1+b)]^{-13}\,(h_a - h_b)}{b + a\,(1+b)}\,.$$

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Problem GA ... center of gravity of better snail

How far from the origin of coordinate system lies the center of gravity of a spiral given in polar coordinates as $r = a\mathrm{e}^{b\varphi}$, where $a = 0.1\,\mathrm{m}$, $b = 1/\pi$ and $\varphi \in (-\infty, 4\pi)$ is the polar angle in radians? Assume the spiral has constant linear density λ. *Lego prefers a logarithmic spiral.*

Setting up the Cartesian coordinates in the plane of the spiral, each of its points can be determined as follows

$$x = a\mathrm{e}^{b\varphi}\cos\varphi\,,$$
$$y = a\mathrm{e}^{b\varphi}\sin\varphi\,.$$

From the Pythagorean theorem, we can express the length of the element of the spiral arc $\mathrm{d}l$ corresponding to angle element $\mathrm{d}\varphi$ as

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = a\mathrm{e}^{b\varphi}\sqrt{(b\cos\varphi - \sin\varphi)^2 + (b\sin\varphi + \cos\varphi)^2}\,\mathrm{d}\varphi = a\mathrm{e}^{b\varphi}\sqrt{1+b^2}\,\mathrm{d}\varphi\,.$$

The mass of the spiral is thus

$$M = \int_{-\infty}^{4\pi} \lambda a\mathrm{e}^{b\varphi}\sqrt{1+b^2}\,\mathrm{d}\varphi = \lambda\frac{a}{b}\mathrm{e}^4\sqrt{1+b^2}\,.$$

Now, we obtain the coordiantes of the center of the gravity straight from the definiton as

$$x_T = \frac{1}{M}\int_0^L x\lambda\,\mathrm{d}l = \frac{1}{M}a^2\lambda\sqrt{1+b^2}\int_{-\infty}^{4\pi}\mathrm{e}^{2b\varphi}\cos\varphi\,\mathrm{d}\varphi = \mathrm{e}^{-4}ba\int_{-\infty}^{4\pi}\mathrm{e}^{2b\varphi}\cos\varphi\,\mathrm{d}\varphi\,,$$
$$y_T = \mathrm{e}^{-4}ba\int_{-\infty}^{4\pi}\mathrm{e}^{2b\varphi}\sin\varphi\,\mathrm{d}\varphi\,.$$

The integrals can be solved numerically, by twice applied per partes, or by using complex exponential function. Either way we get

$$x_T = \mathrm{e}^{-4}ba\left[\mathrm{e}^{2b\varphi}\frac{2b\cos\varphi + \sin\varphi}{4b^2+1}\right]_{-\infty}^{4\pi} = \frac{2\mathrm{e}^4b^2a}{4b^2+1} \doteq 0.79\,\mathrm{m}\,,$$
$$y_T = \mathrm{e}^{-4}ba\left[\mathrm{e}^{2b\varphi}\frac{2b\sin\varphi - \cos\varphi}{4b^2+1}\right]_{-\infty}^{4\pi} = \frac{-\mathrm{e}^4ba}{4b^2+1} \doteq -1.24\,\mathrm{m}\,.$$

The last thing we need is to find the distance from the origin of coordinates. Clearly from the Pythagorean theorem, it is $\sqrt{x_T^2 + y_T^2} \doteq 1.5\,\mathrm{m}$.

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Problem GB ... falling tree

Jáchym cut down a tree, which we can approximate with a thin homogeneous cylinder of height 11 m. *The tree is still partially connected to the stump, so its top describes the arc of the circle. At what time does the top of the tree hit the ground if we start the stopwatch when the tree passes deviation* 1°*?* *Jáchym gives solvers a hard time.*

We will proceed from the law of conservation of energy. Initially, the tree with the length l has the center of gravity at height $\frac{l}{2}$. Therefore, its potential energy is

$$E_{\mathrm{p}0} = mg\frac{l}{2}\,.$$

We denote polar angle (i.e., the angle of tree deviation from vertical axis) during the fall by φ. We can express potential energy of tree as a function of angle φ as

$$E_{\mathrm{p}} = mg\frac{l}{2}\cos\varphi\,.$$

Since the energy is conserved, we can determine the corresponding angular speed. We will approximate the tree by a thin homogeneous rod with the moment of inertia relative to its end being equal to $J = \frac{1}{3}ml^2$. Hence, we can compute the kinetic energy as

$$E_{\mathrm{k}} = \frac{1}{2}J\omega^2 = \frac{1}{2}\frac{1}{3}ml^2\omega^2 = E_{\mathrm{p}0} - E_{\mathrm{p}} = mg\frac{l}{2}\left(1-\cos\varphi\right)\,,$$

where $\omega = \dot{\varphi}$ is the angular speed of the tree.

The usage of the angular speed is convenient as it satisfies expression

$$\omega\,\mathrm{d}t = \mathrm{d}\varphi \quad\Rightarrow\quad \mathrm{d}t = \frac{\mathrm{d}\varphi}{\omega}\,.$$

If we express the angular speed ω as a function of angle φ and integrate it, we obtain the total fall time as a function of φ, i.e.,

$$t = \int_{\varphi_0}^{\varphi_1}\frac{\mathrm{d}\varphi}{\omega}\,,$$

where φ_0 and φ_1 are the initial and final angles, between which we measure the time. After substitution for ω from energy conservation (above), we obtain

$$t = \sqrt{\frac{l}{3g}}\int_{\varphi_0}^{\varphi_1}\frac{\mathrm{d}\varphi}{\sqrt{1-\cos\varphi}}\,.$$

Firstly, we will try to compute the antiderivative when we do not have to consider boundaries.

To get rid of the square root in the denominator, we try substitution $\frac{\varphi}{2} = \psi$, providing

$$2\int\frac{\mathrm{d}\psi}{\sqrt{1-\cos 2\psi}} = 2\int\frac{\mathrm{d}\psi}{\sqrt{1-\cos^2\psi+\sin^2\psi}} = \sqrt{2}\int\frac{\mathrm{d}\psi}{\sin\psi}\,.$$

We will try to halve the angle once again, i.e., $\theta = \frac{\psi}{2}$. Then

$$\sqrt{2}\int \frac{\mathrm{d}\theta}{\sin\theta\cos\theta}\,.$$

Since we obtained the product of sine and cosine of the integrated variable, we will use common trigonometric substitution for this case, $u = \tan\theta$, therefore,

$$\mathrm{d}u = \frac{1}{\cos^2\theta}\,\mathrm{d}\theta\,,$$

from which we get

$$\sqrt{2}\int \frac{\cos\theta}{\sin\theta}\,\mathrm{d}u = \sqrt{2}\int \frac{1}{u}\,\mathrm{d}u\,.$$

Finally, we obtained the expression we can easily integrate, providing

$$\int \frac{\mathrm{d}\varphi}{\sqrt{1-\cos\varphi}} = \sqrt{2}\int\frac{\mathrm{d}u}{u} = \sqrt{2}\ln u + C = \sqrt{2}\ln\tan\theta + C = \sqrt{2}\ln\tan\frac{\psi}{2} + C = \sqrt{2}\ln\tan\frac{\varphi}{4} + C\,.$$

Thus, the fall of the tree with the initial deviation 1° to the horizontal position takes

$$t = \sqrt{\frac{2l}{3g}}\left[\ln\tan\frac{\varphi}{4}\right]_{1^\circ}^{90^\circ} \doteq 6.44\sqrt{\frac{l}{3g}} = 3.9\,\mathrm{s}\,.$$

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