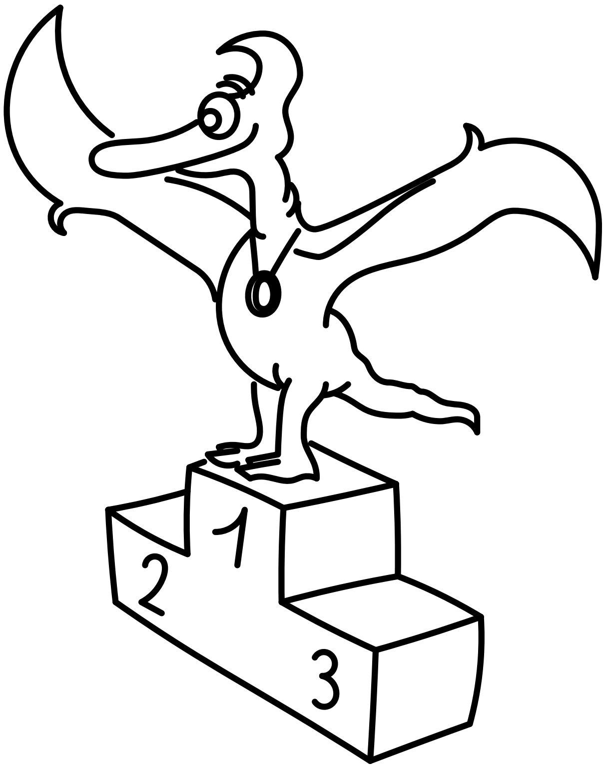
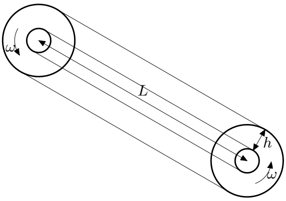
13^{th} FYKOS Physics Brawl
Solutions of problems



Problem AA ... escalator strangeness



You may have noticed that the escalator belt for holding frequently moves at a different speed than the stairs themselves. Consider a simplified model where the belt and the stairs are attached to (wrapped around, without slipping) cylinders with a common axis at each end of the escalator. The cylinders are rotating with a constant angular velocity $\omega = 0.36\,\mathrm{rad{\cdot}s^{-1}}$ *and the radii of inner cylinders are* $r_1 = 1.2\,\mathrm{m}$*. Find the ratio of the velocity of the belt to the velocity of the stairs. The escalator in the model has length* $l = 32\,\mathrm{m}$ *and the distance from the stairs to the belt is* $h = 0.6\,\mathrm{m}$*.*

The radii of the cylinders on which the belt is rotating are $r_2 = r_1 + h$. The relation between linear and angular velocity $v = \omega r$ gives the ratio in question

$$\frac{v_2}{v_1} = \frac{\omega r_2}{\omega r_1} = \frac{r_1 + h}{r_1} = \frac{3}{2}\,.$$

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Problem AB ... far from the planet

How much (in percent) do airliners prolong their path by flying at height $H = 11\,\mathrm{km}$*, compared to the distance flown just above the sea level? Neglect the travel distance necessary to reach a particular height. Assume that the radius of Earth is* $R_\mathrm{Z} = 6{,}373\,\mathrm{km}$*.*

Karel was thinking about air traffic.

If we want to fly from point A to point B and compare the distances flown at zero height and at height H, we don't even need to know the ground distance of these points. The aeroplane moves by the same angle ϑ with respect to the centre of the Earth. Therefore, we may write the ratio of the distance s_H flown at height H to s_0 at sea level as

$$K = \frac{s_H}{s_0} = \frac{(R_\mathrm{Z} + H)\,\vartheta}{R_\mathrm{Z}\vartheta} = \frac{R_\mathrm{Z} + H}{R_\mathrm{Z}} = 1 + \frac{H}{R_\mathrm{Z}} \doteq 1.0017\,.$$

Now we have the ratio. The question is about the relative increase in travel distance. We can find this by subtracting 1 from K, since the travel distance is longer by $\Delta = K - 1 = 0.17\,\%$. Compared to other effects that may change fuel consumption or flight length, this is clearly negligible.

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Problem AC ... it sparks

Find the distance d *between an electric line and a locomotive's pantograph (traction unit) which is needed for electric discharge (a spark) to occur. The electric line uses direct current and its voltage with respect to the rail (the ground) is* $U = 3\,\mathrm{kV}$*. The dielectric strength of air (the intensity of the electric field between electrodes needed for a spark) is* $E = 3.0\cdot 10^{6}\,\mathrm{kg{\cdot}m{\cdot}A^{-1}{\cdot}s^{-3}}$*. Assume that the relative permittivity and permeability of air are both 1.*

Dodo was thinking after a long journey home.

The intensity of a homogeneous field E is related to voltage difference by the formula

$$U = Ed\,,$$

where d is the distance between the points where we measure voltage. In this case, the electric field isn't homogeneous, but we may approximate it as a homogeneous field with the same average intensity E, since d will most likely be very small. Then, we can use this formula to obtain

$$d = \frac{U}{E} = 1\,\mathrm{mm}\,.$$

We can see that d is indeed quite small, so our approximation should be accurate.

As an afternote, consider how much more complicated this problem could get if we wanted an exact answer. First of all, we need to know the exact geometry of the pantograph (and the electric line); of course, a locomotive is moving, which induces additional motion of charge on the electric line and a general electromagnetic field. The electric intensity isn't usually defined as an average either, but for a homogeneous field. It's not even clear what kind of average it should be, precisely because there are so many factors to consider - in this case, we're lucky to have a good approximation.

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Problem AD ... beware tram

We are standing next to a straight tram track. We see a tram approaching from the right with a constant velocity v and another tram approaching from the left with a constant velocity u. In the front window of the second tram, we glimpse the reflection of the first tram. Find the velocity with which the reflection is approaching us.

Dodo was bored on the way to climbing class.

The front window of the second tram is approaching us with the velocity u. If we were watching our reflection, it would obviously be approaching with velocity $2u$ and therefore, the velocity of the reflection of the first tram with respect to us is $2u + v$.

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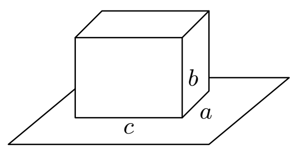
Problem AE ... unstable Danka

Consider a cuboid with dimensions $a = 20\,\mathrm{cm}$, $b = 30\,\mathrm{cm}$, $c = 50\,\mathrm{cm}$ and density $\varrho = 620\,\mathrm{kg{\cdot}m^{-3}}$. The cuboid is lying on a horizontal surface in a homogeneous gravitational field in such a way that the faces with dimensions a and c are facing up/down. What is the stability of the cuboid, i.e. the minimum amount of energy necessary to overturn it? *Danka was dropping stuff.*



We can see that it's more efficient to turn the cuboid over one of the edges with length c, because a is shorter, so it's easier to lift the center of mass so that the cuboid would turn over the edge with length c.

In order to turn the cuboid over, we have to move the centre of mass in such a way that it's directly above the edge with height c and push it a bit further. This change in the position of the centre of mass requires increasing the potential energy. We can calculate the stability as the maximum increase in potential energy of the centre of mass. Let's assume that potential is zero on the surface. At the beginning, the cuboid has potential energy

$$E_{p1} = \varrho abcg\frac{b}{2}\,.$$

When the center of mass is right above edge c, its height h is

$$h = \frac{1}{2}\sqrt{a^2 + b^2}$$

and its potential energy is

$$E_{p2} = \varrho abch\,.$$

The stability of the cuboid is the difference between E_{p2} and E_{p1}, which is

$$\Delta E = E_{p2} - E_{p1}\,,$$

$$\Delta E = \frac{1}{2}\varrho gabc(\sqrt{a^2 + b^2} - b)\,,$$

$$\Delta E \doteq 5.52\,\mathrm{J}\,.$$

The stability of the cuboid is 5.52 J.

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Problem AF ... flat-Earth

It is well-known that the Earth is flat and supported by elephants. Above it, there is a dome from which the Sun, the Moon and stars are hanging. If the Moon with mass $7.348 \cdot 10^{22}$ kg was hanging from a steel rope with a uniform circular cross-section, what would be the minimum diameter of its cross-section if we wanted to keep a 10 % reserve (the rope should not break if the mass increased by 10 %)? The yield stress of steel is 700 MPa. The Moon is placed in uniform gravity g. The weight of the rope can be neglected.

Mikuláš has a weakness for Terry, Terry Pratchett.

The solution can be found by combining formulas for stress $\sigma = F/S$, force due to gravity $F = mg$ and area of a circle $S = \pi d^2/4$. We obtain the formula

$$\sigma_{\mathrm{max}} = \frac{4 \cdot 1.1mg}{\pi d^2}$$

and from it, we can express the diameter d as

$$d = 2\sqrt{\frac{1.1\,mg}{\pi\sigma_{\mathrm{max}}}}\,.$$

Plugging in the numerical values, we get $d = 38{,}000\,\mathrm{km}$. This is several times larger not only than the diameter of the Moon, but also of the Earth, so we can conclude that an explanation of celestial mechanics using massive rocky spheres hurtling across space cannot be completely excluded.

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Problem AG ... the Prague Metro

You are running on an escalator. The time it takes you to run from the bottom to the top when the escalator is turned on is t_z. If the escalator is turned off, the time it takes you is t_v ($t_\mathrm{v} < t_\mathrm{z}$). Your running speed is v_c. Find the speed of the escalator.

Legolas was running on escalators in the underground.

We can determine the length of the escalator l_e using the time t_v as

$$l_\mathrm{e} = \frac{1}{2} v_\mathrm{c} t_\mathrm{v} \,,$$

because it's crossed twice (upstairs and downstairs). If t_1 is the time it takes to get upstairs and t_2 the time it takes to get downstairs when the escalator is turned on, we get

$$t_\mathrm{z} = t_1 + t_2 \,,$$

$$t_\mathrm{z} = \frac{l_\mathrm{e}}{v_\mathrm{c} + v_\mathrm{e}} + \frac{l_\mathrm{e}}{v_\mathrm{c} - v_\mathrm{e}} \,,$$

$$\frac{t_\mathrm{z}}{v_\mathrm{c} t_\mathrm{v}} = \frac{v_\mathrm{c}}{v_\mathrm{c}^2 - v_\mathrm{e}^2} \,,$$

from which we obtain the speed of the escalator

$$v_\mathrm{e} = v_\mathrm{c} \sqrt{1 - \frac{t_\mathrm{v}}{t_\mathrm{z}}} \,.$$

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Problem AH ... prefixious

Sort the first eight letters of the English alphabet according to their use as SI prefixes in descending order of size. Make sure to use correct capitalisation of letters. In case some letter is not used as a prefix, do not use it. In case both forms (lowercase and uppercase) are, use both.

Dodo is studying the alphabet.

From the letters a, b, c, d, e, f, g, h, A, B, C, D, E, F, G, H, only a, c, d, E, f, G, h are SI prefixes, with the following names and meanings: atto $= \cdot 10^{-18}$, centi $= \cdot 10^{-2}$, deci $= \cdot 10^{-1}$, exa $= \cdot 10^{+18}$, femto $= \cdot 10^{-15}$, giga $= \cdot 10^{9}$, hecto $= \cdot 10^{2}$. The correct order is E, G, h, d, c, f, a.

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Problem BA ... landing accident

Some flight navigation instruments are based on air pressure. Let's focus on the altimeter, which displays the altitude of the plane calculated using the static pressure of surrounding air. For this device to work correctly, it is necessary on every airport to recalibrate it with the current air pressure at sea level, calculated from local air pressure. Near the sea level, we can then use the approximation that ascent by $\Delta h = 8.0\,\mathrm{m}$ corresponds to a pressure drop $\Delta p = 1.0\,\mathrm{hPa}$. If a pilot accidentally recalibrates the device with sea level pressure $p_\mathrm{err} = 1{,}021\,\mathrm{hPa}$ instead of the correct $p_\mathrm{r} = 1{,}012\,\mathrm{hPa}$, how high above the sea level would the airplane be when the altimeter shows $450\,\mathrm{m}$?

Karel was learning about the importance of the QHN.

We can write the ratio of altitude change to pressure change as

$$K = -\frac{\Delta h}{\Delta p} = -8.0\,\mathrm{m{\cdot}hPa^{-1}}\,.$$

The difference between the actual pressure and pressure set in the altimeter is $\Delta P = p_\mathrm{r} - p_\mathrm{err} = -9\,\mathrm{hPa}$. This corresponds to the altimeter "thinking" the altitude is greater by

$$H = K\Delta P = \frac{\Delta h}{\Delta p}\left(p_\mathrm{r} - p_\mathrm{err}\right) = 72\,\mathrm{m}\,.$$

If the instrument shows 450 m, the airplane is actually 378 m above the sea level. This is quite a big difference, for example if the airplane is approaching the Prague - Ruzyne airport, which is at the altitude 380 m. We could say it's a difference between life and death.

If you're confused by the sign convention and whether it determines that the real altitude is higher or lower than the displayed altitude, think about which pressures are used. Pressure always decreases with increasing altitude. If the pressure set in the altimeter is higher than real pressure, it seems that the air column below us is greater. That's because our altimeter measures the surrounding pressure and compares it to the set pressure, which it assumes to be the pressure at sea level. It seems that the pressure drop from sea level is greater than it actually is.

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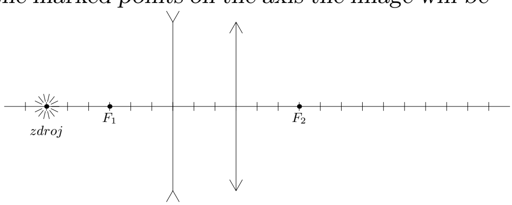
Problem BB ... drawing

In the picture, you can see an optical axis, on which a point source of light ("zdroj"), a focus point of the first (diverging) lens, a thin diverging lens, a thin converging lens, and a focus point of the second (converging) lens are marked. Each distance is equal to three units (of distance). Draw into the picture the position of the source's image after it is formed by both lenses. Your task is only to decide between which two of the marked points on the axis the image will be formed. You may solve it by an arbitrary method.

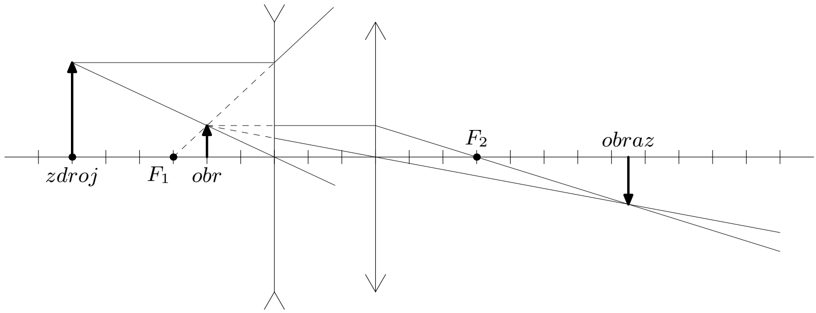


Matěj told himself it would be nice to draw something during the competition.

The following is a solution by image.

We can also measure each distance (the source is placed 6 units of distance before the lens and the focal lengths are -3 and 3) and using the thin lens equation

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{f}\,,$$



we can find out that the image will be formed 2 units of distance before the first lens ("obr"). This image will be formed 5 units of distance from the second lens, which will form it at the distance 7.5 units before itself ("obraz").

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Problem BC ... acceleration from planets

Consider a system of two homogeneous spherical planets, both with densities $\varrho = 4{,}200\,\mathrm{kg{\cdot}m^{-3}}$. *The smaller planet has radius* $R = 4.20 \cdot 10^6\,\mathrm{m}$ *and the larger one has radius* $R_2 = 2R$. *The distance of their centres is* $r = 16R$. *What is the gravitational acceleration (including its direction) acting on a dust particle located at a distance* $x = 8R$ *from the centres of both planets?* *Karel likes to play with impossible planetary systems.*

The gravitational acceleration that acts on the dust particle outside a spherically symmetrical body at a distance x from its centre is

$$a_g = G\frac{m}{x^2}\,,$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$ is the gravitational constant and m is the mass of the body which imparts this acceleration.

In this problem, there are two spherically symmetrical bodies and the only relevant difference between them is in their masses. The mass of the smaller sphere is

$$m_1 = \varrho V_1 = \frac{4}{3}\pi\varrho R^3\,.$$

Mass is proportional to the cube of radius and the second sphere's radius is twice as large as the first sphere's, so its mass is $m_2 = 8m_1$.

Gravitational acceleration always acts towards the body which causes it. If we place the spheres and dust particle on an axis in such a way that the lighter sphere is on the left of the particle and the heavier one on the right, the principle of superposition gives the total acceleration (pointing to the right)

$$a = a_1 + a_2 = -G\frac{m_1}{x^2} + G\frac{m_2}{x^2} = \frac{7Gm_1}{x^2} = \frac{7\pi G\varrho R}{48} \doteq 0.54\,\mathrm{m{\cdot}s^{-2}}\,.$$

The total gravitational acceleration acting on the dust particle located exactly between the centres of the planets is $0.54\,\mathrm{m\cdot s^{-2}}$. Its direction is away from the centre of the smaller planet, towards the centre of the larger planet.

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Problem BD ... drinking tea

Consider a cylindrical container with height h and base area S, filled with tea with density ϱ. There is a little hole with area $s \ll S$ at the bottom of the container. This hole is used to attach a horizontal tube with a faucet and a manometer (a vertical tube, open at the top) to the container. Find the formula describing the height of liquid in the manometer $l(t)$ over time after the faucet is opened. Assume that this height adapts instantly to changes in the flow and that tea is an ideal liquid (incompressible, with zero viscosity and surface tension).

Dodo was thinking in the canteen and forgot to eat.

The velocity of the fluid flowing through the hole in the container can be expressed as $v = \sqrt{2gh}$. The overpressure at the bottom of the manometer p (with respect to atmospheric pressure) follows from the Bernoulli equation:

$$p = \varrho g h - \frac{1}{2}\varrho v^2 = 0\,\mathrm{Pa}\,.$$

This conclusion could have also been made after realising that for an ideal liquid, there is no difference between the pressure at the bottom of the manometer and the pressure just above the hole through which the water flows.

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Problem BE ... at the bottom of the lake

How much higher is the pressure at the bottom of the Dead Sea as a result of it being salty? We are interested in the difference with respect to a hypothetical situation when the sea would be sweet. Consider a situation where the lake's surface (the Dead Sea is a lake, which is drying out continuously) is $h_0 = 430\,\mathrm{m}$ under the sea level, and the deepest point of the lake is $\Delta h = 298\,\mathrm{m}$ lower. The density of sweetwater is $\varrho_0 = 997\,\mathrm{kg\cdot m^{-3}}$, while the current density of the lake is $\varrho = 1{,}240\,\mathrm{kg\cdot m^{-3}}$. Assume that the depth of the lake after demineralizing would be the same.

Karel was learning something about air pressure.

We do not need data about how deep the water level of The Dead Sea is, because we can consider that above the water level, the atmosphere is the same in both cases and pressure changes only in the water column.

We can consider the depth of the lake after demineralizing to be the same, so the difference of pressures at the bottom of the lake is given by the difference of hydrostatic pressure in saltwater $p_1 = \varrho \Delta h g$ and in sweetwater $p_2 = \varrho_0 \Delta h g$, where g is the acceleration due to gravity. We get the pressure difference

$$\Delta p = p_1 - p_2 = (\varrho - \varrho_0)\,\Delta h g \doteq 710\,\mathrm{kPa}\,.$$

At the bottom of the Dead Sea nowadays, the pressure is about 710 kPa higher than if the water in it was sweet (distilled). The difference is almost the same as 7 atmospheres. It is true that we considered the density of completely clear water, but sweetwater in nature contains some soluble substances and because of that, the real difference should be lower.

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Problem BF ... take a guess

By how many orders of magnitude is the volume of the observable universe greater than the volume of a hydrogen atom? In other words, find the order of magnitude of the ratio (volume of observable universe) / (volume of a hydrogen atom). The observable universe has diameter 28.5 Gpc. *The van der Waals radius of hydrogen is* $r = 1.2\,\text{Å}$.

Dodo did something wrong once again.

We can convert the diameter of the observable universe to its radius $R = 4.41 \cdot 10^{26}\,\mathrm{m}$. One Angstrom is $1\,\text{Å} = 1 \cdot 10^{-10}\,\mathrm{m}$, so the ratio of radii is $R/r = 3.7 \cdot 10^{36}$. Taking the decimal logarithm, we get $\log \frac{r}{r_0} = 36.56$. The ratio of volumes is

$$\log \frac{V}{V_0} = 3 \log \frac{r}{r_0} = 109.68 \approx 110\,.$$

The volume of the observable universe is bigger by 110 orders of magnitude.

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Problem BG ... exosystem with a planet

Imagine that we discovered another planetary system, which is similar to ours. The local sun has mass $0.900 M_\mathrm{S}$ *(a multiple of the mass of our Sun). If the year on a local planet similar to our Earth has the same length as our year, at what distance from the local sun would it have to be? For simplicity, assume that the planet has a circular orbit. Compute the result as a multiple of* au, *i.e. the middle distance between the centers of the Earth and the Sun.*

Karel was thinking about exoplanets.

We assume that the planet orbits on a circle, so the centripetal force F_d has to be equal to gravitational force F_g all the time. Other forces are not appearing in this task (we neglect the effects of other planets because we do not know where they are located) and the planet orbits its sun uniformly (with velocity v). Then we get

$$F_\mathrm{d} = F_\mathrm{g} \qquad \Rightarrow \qquad m\frac{v^2}{r} = G\frac{mM}{r^2} \qquad \Rightarrow \qquad v^2 = G\frac{M}{r}\,,$$

where m is the mass of the planet, M is the mass of the local sun, G is the gravitational constant and r is the distance between the planet and the center of the local sun. Now we use the formula $v = 2\pi r/T$, i.e. the rotational speed of the planet corresponds to the fact that it makes one revolution around its sun in one year (time T). We get a relation between r and other parameters

$$\frac{4\pi^2 r^2}{T^2} = G\frac{M}{r} \qquad \Rightarrow \qquad r = \sqrt[3]{\frac{GM}{4\pi^2}T^2}\,.$$

Now we see that the orbital radius depends on gravitational acceleration, which is constant, other constants and on the mass of the sun and orbital time of the planet. Because we have the same duration of one year and a lower mass of the sun, it is enough to plug these values into the formula for r and we get the orbital radius as a multiple of the Earth's orbital radius r_Z

$$r = \sqrt[3]{G\frac{0.900\,M}{4\pi^2}T^2} \doteq 0.965\,r_Z\,.$$

The radius of the planet's orbit in the discovered solar system has to be 0.965 au. With a decrease of the mass of the sun by about 10 %, the orbital radius has to decrease by about 3.5 %.

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Problem BH ... price of a flight

We are flying in an aircraft-carrier BAe 146-300/RJ100 and we want to know whether it would be cheaper to fly at low or high velocity, when we want to fly $s = 250\,\mathrm{km}$ at the flight level FL 310. The aircraft at this flight level has fuel consumption $M_h = 2{,}517\,\mathrm{kg{\cdot}h^{-1}}$ at velocity $v_h = 429\,\mathrm{kn}$ and $M_l = 1{,}724\,\mathrm{kg{\cdot}h^{-1}}$ at velocity $v_l = 377\,\mathrm{kn}$. For which type of flight is the fuel consumption lower and how much fuel (in kg) do we save at the distance s? One knot (kn) is $1{,}852\,\mathrm{m{\cdot}h^{-1}}$. *Karel is at ATCo training.*

We start with converting the units of velocity to the units we need, so $v_h = 429\,\mathrm{kn} \doteq 794.5\,\mathrm{km{\cdot}h^{-1}}$ and $v_l = 377\,\mathrm{kn} \doteq 698.2\,\mathrm{km{\cdot}h^{-1}}$. As we are given the fuel consumption in kilograms per hour, we have to determine the flight length for each type of flight,

$$t_h = \frac{s}{v_h} \doteq 0.315\,\mathrm{h}\,, \quad t_l = \frac{s}{v_l} \doteq 0.358\,\mathrm{h}\,.$$

The total amount of consumed fuel for each type of flight is

$$m_h = M_h t_h = M_h\frac{s}{v_h} \doteq 792\,\mathrm{kg}\,, \quad m_l = M_l t_l = M_l\frac{s}{v_l} \doteq 617\,\mathrm{kg}\,.$$

Maybe it is suprising, but we found out that if we fly slower, the fuel consumption is lower - at least on flight level 310 (31,000 feet above the sea level, around 9.3 km). The answer is that slower flight is more economical and we save around 175 kg of fuel.

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Problem CA ... when physicist cooks

In the manual for Legolas' kettle, it is written that $U_0 = 240V$, $P_0 = 2200W$ (i.e. the kettle consumes electrical power P_0 if the voltage is U_0). He poured exactly $V = 1.5\,\mathrm{l}$ of water with temperature $T_1 = 20\,^\circ\mathrm{C}$ into the kettle. Then he switched the kettle on and heated the water to a temperature $T_2 = 80\,^\circ\mathrm{C}$. His stopwatch showed that the process lasted for $t = 4.2\,\mathrm{min}$. What voltage does Legolas have in his power socket? Assume that the resistance of the kettle is independent on temperature and the efficiency of heating water is $\eta = 90\,\%$.

Lego often cooks. . . tea.

First, we calculate that the water absorbed heat

$$Q = cm\Delta T = cV\varrho\left(T_2 - T_1\right) .$$

From this, we can find the average power consumption of the kettle

$$P_\mathrm{k} = \frac{1}{\eta}\frac{Q}{t} = \frac{cV\varrho\left(T_2 - T_1\right)}{\eta t} .$$

Now we use the information from the manual. We know that the kettle consumes power P_0 at voltage U_0. We can express the resistance of the kettle or we can simply realize that the electrical power consumption is proportional to the square of voltage, which gives

$$\frac{P_\mathrm{k}}{P_0} = \frac{U_\mathrm{k}^2}{U_0^2} .$$

Therefore, the result is

$$U_\mathrm{k} = U_0\sqrt{\frac{cV\varrho\left(T_2 - T_1\right)}{\eta t P_0}} .$$

Plugging in the numerical values, we get $U_\mathrm{k} \approx 208\,\mathrm{V}$.

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Problem CB ... the price of charging a car battery

How much does it cost to fully charge an empty battery ($12\,\mathrm{V}$, $60\,\mathrm{Ah}$)*? We will use a battery charger with parameters* $12\,\mathrm{V}$, $4.2\,\mathrm{A}$ *and effectivness* $72\,\%$. *The price for electricity is* $k = 4.0\,\text{Kč}\cdot\mathrm{kWh}^{-1}$. *Karel. Don't even ask.*

To obtain the price, we need to specify the energy E_0 that we need to supply the battery. This is given by the product of the battery voltage $U = 12\,\mathrm{V}$ and her capacity $Q = 60\,\mathrm{Ah}$

$$E_0 = QU = 720\,\mathrm{Wh} .$$

In order to determine the energy that we need from the power grid, we need to consider the efficiency of the charger

$$E = \frac{E_0}{\eta} = \frac{QU}{\eta} = 1.0\,\mathrm{kWh} .$$

The final price x which we have to pay for is then equal to

$$x = Ek = \frac{QU}{\eta}k = 4.0\,\text{Kč} .$$

One car battery charging will then cost $4.0\,\text{Kč}$. Note that although it is said that the car battery has a voltage of $12V$, it is not Exactly constant and in fact, the voltage on battery is slightly higher. Especially in a charged state. That would increase the price a little bit.

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Problem CC ... mirrors

Consider almost perfectly polished plane mirrors and set them parallel to a distance of 5 m*. When Matej looked in one of them, he did not see the infinity of reflections. How far away is the last visible image if one reflection reflects only* 98.5 % *light? He is able to detect an image that has* 10 % *of the original brightness. Do not consider the dimensions of his head or other loss of brightness.*

The beam of light has to reflect $\log_{0.985}(0.1) = 152$ times, before its intensity drops below 10 %. Each reflection corresponds to five meters of distance and since the number is even, the distance before the first reflection and the distance after the last reflection sum up to 5 m. Thus, the farthest image will be seen at a distance $152 \cdot 5\,\mathrm{m} \doteq 760\,\mathrm{m}$.

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Problem CD ... model of an atom

An electron in the Bohr model of the hydrogen atom is orbiting around fixed proton on a circle with radius r. Its angular momentum is quantized, so it obtains values $L = n\hbar$, where n is a number describing quantum state of the electron. What is the radius r of the circular orbit of the electron, when it is present in the second quantum state?

Danka was reminiscing about the Physics Olympiad.

Let's denote $m = 9.109 \cdot 10^{-31}\,\mathrm{kg}$ as a mass of the electron and v as its velocity. The electron orbiting on a circle around the proton is affected by centrifugal force

$$F_\mathrm{o} = \frac{mv^2}{r}$$

and the electric attractive force

$$F_\mathrm{e} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\,,$$

where e is elementary charge. These forces are compensated by each other, so we get

$$F_\mathrm{o} = F_\mathrm{e}\,,$$

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\,.$$

The angular momentum of the electron is

$$L = n\hbar = mvr\,,$$

from where we express the velocity of the electron

$$v = \frac{n\hbar}{mr}\,.$$

We induct this expression to the equation acquired from the equilibrium of forces and we express the radius

$$r = \frac{4\pi\varepsilon_0\hbar^2n^2}{me^2}\,.$$

In our case $n = 2$, so after evaluating the result numerically we get $r \doteq 2.1 \cdot 10^{-10}\,\mathrm{m}$.

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Problem CE ... degradation of the efficiency of a kettle

Consider a kettle with a power $P = 1{,}800\,\mathrm{W}$ *which heats* $m = 1.2\,\mathrm{kg}$ *of water from temperature* $t_0 = 18\,^{\circ}\mathrm{C}$ *to* $t_1 = 100\,^{\circ}\mathrm{C}$ *in* $\tau = 4.2\,\mathrm{min}$. *Unfortunately, it takes another* $\Delta\tau = 15\,\mathrm{s}$ *for it to "realise" the water has boiled and to turn off. By how much does the efficiency of the kettle as a heat engine decreases as opposed to it turning off right after the water has boiled? Suppose that the heat capacity of water is constant,* $c = 4{,}200\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. *Karel was hypnotizing a kettle.*

The heat necessary for the water to boil is given by $Q = mc\,(t_1 - t_0) \doteq 413\,\mathrm{kJ}$. The kettle's energy consumption for that period τ is $E_0 = P\tau = 454\,\mathrm{kJ}$ and consequentially its efficiency will be

$$\eta_0 = \frac{Q}{E_0} = \frac{mc\,(t_1 - t_0)}{P\tau} \doteq 91.1\,\%\,.$$

With a similar reasoning, we can figure out the energy consumption for the period $\tau + \Delta\tau$ as $E_1 = P\,(\tau + \Delta\tau) \doteq 481\,\mathrm{kJ}$. The corresponding efficiency then follows:

$$\eta_1 = \frac{Q}{E_1} = \frac{mc\,(t_1 - t_0)}{P\,(\tau + \Delta\tau)} \doteq 86.0\,\%\,.$$

Since the task is to calculate the difference in efficiency, we can easily conclude

$$\Delta\eta = \eta_1 - \eta_0 = \frac{mc\,(t_1 - t_0)\,\Delta\tau}{P\tau\,(\tau + \Delta\tau)} \doteq 5.1\,\%\,.$$

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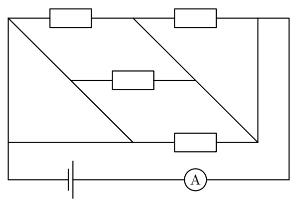
Problem CF ... zkkzgt zjazq

A bomb was found at a field. The countdown ends at 13:30, so the poor farmers cannot escape the explosion's deadly radius anymore. At which point(s) should you cut the circuit if you know that the bomb does not explode only when the ammeter shows exactly $15\,\mathrm{mA}$ *at 13:30? The circuit is powered by four* $1.5\,\mathrm{V}$ *batteries and each resistor has resistance* $1\,\mathrm{k\Omega}$. *Mark the point(s) to cut in the figure. Matěj is surely not a terrorist.*



Ohm's law says that the circuit needs to have total resistance $\frac{2}{5}R$, where R is the resistance of one resistor. If there are only four available resistors, the only way to achieve that is by connecting in parallel: one resistor, one resistor and two serially connected resistors. That corresponds to exactly one cut, as shown in the figure.

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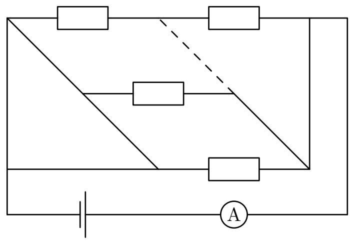
Problem CG ... playing with marbles

Danka has a marble with radius $R = 1.0\,\mathrm{cm}$ *and mass* $m = 20\,\mathrm{g}$. *She pushes the marble in such a way that a horizontal force* $F = 15\,\mathrm{N}$ *acts on it for a time* $t = 0.2\,\mathrm{s}$ *(we neglect the rolling*



friction during this time period and the marble does not slip). How long does the bead move? The rolling resistance coefficient is $\xi = 0.03m$. Assume that the bead does not slip.

Danka found a marble.

The marble gains momentum p,

$$p = Ft = mv_0 \,,$$

where v_0 is the velocity of the marble immediately after it is pushed. There is a constant force of rolling resistance acting on the marble,

$$T = \frac{\xi mg}{R} \,.$$

The marble is moving with uniformly accelerated (decelerated) motion, where the acceleration is given by Newton's 2nd law

$$a = -\frac{T}{m} = -\frac{\xi g}{R} \,.$$

Let's denote the answer by t_v. Then,

$$v_0 - at_{\mathrm{v}} = 0 \,.$$

Substituting from the previous formulas, we can express t_v

$$0 = \frac{Ft}{m} - \frac{\xi g}{R} t_{\mathrm{v}} \,,$$

$$t_{\mathrm{v}} = \frac{FtR}{\xi mg} \doteq 5.1\,\mathrm{s} \,.$$

The marble remains in motion for 5.1 s.

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Problem CH ... flat-Earth reloaded

It is well-known that the Earth is flat and supported by elephants. Above it, there is a dome from which the Sun, the Moon and stars are hanging. If the Moon with mass $7.348\cdot 10^{22}$ kg was hanging from a steel rope with a uniform circular cross-section, what would be the minimum radius of its cross-section? The length of the rope is $3.2\,\mathrm{km}$, the yield stress of steel is $700\,\mathrm{MPa}$ and its density is $7{,}850\,\mathrm{kg{\cdot}m^{-3}}$. The Moon and the rope are placed in uniform gravity g. Neglect the mutual gravitational influence of the rope and the Moon.

Jáchym has a weakness for reloaded problems.

The stress in the rope is maximum at its top, since the stress in it isn't caused only by gravity acting on the Moon, but also gravity acting on the whole rope. Let's denote the mass of the Moon by m the density of the rope by ϱ, its radius by r and its length by l. The total force acting at the top of the rope due to gravity is

$$F_g = \left(m + \pi r^2 l \varrho\right) g \,.$$

The given yield stress σ, multiplied by the cross-sectional area of the rope πr^2 gives the maximum force which may act on the rope

$$F_{\mathrm{max}} = \sigma \pi r^2 \,.$$

Finally, we can just compare the two forces and express the radius

$$F_g = F_{\mathrm{max}} \,,$$
$$r = \sqrt{\frac{mg}{\pi\left(\sigma - l\varrho g\right)}} \doteq 2.25 \cdot 10^7\,\mathrm{m} \,.$$

Note that this value is very unrealistic and construction of such a rope wouldn't make much sense overall.

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Problem DA ... not even an oil stain

There is a standardized light source with luminous intensity $I_0 = 10\,\mathrm{kcd}$ and a light bulb; the distance between them is $d = 3\,\mathrm{m}$. If we place a sheet of paper with an oil stain between the sources at a distance $l = 1\,\mathrm{m}$ from the light bulb, the whole sheet appears equally bright. Find the luminous intensity I of the light bulb.

Dodo was reminiscing about a discontinued lab experiment.

The oil stain becomes invisible when the illuminance is the same on both sides

$$\frac{I}{l^2} = F = \frac{I_0}{(d - l)^2} \,.$$

From this, we obtain the luminous intensity of light bulb $I = I_0 \left(d/l - 1\right)^{-2} = 2.5\,\mathrm{kcd}$.

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Problem DB ... rabbit no racist

A small, fat, spherical magic rabbit with diameter $d = 30\,\mathrm{cm}$ is floating in space near Earth. The rabbit is cold, so it decides to paint itself black with some Vanta Black colour, which reflects only $\eta = 0.035$ of incident radiation. What is the power it absorbs from solar radiation? The power radiated by the Sun is $W = 3.8 \cdot 10^{26}\,\mathrm{W}$. *Honza was cold.*

We may consider the distance of the rabbit from the Sun to be $r = 1.50 \cdot 10^{11}$ m. Since the rabbit-sphere absorbs basically all sunlight, we don't need to consider angle of incidence. The absorbed power is

$$P = (1-\eta)\, W \frac{\pi\, (d/2)^2}{4\pi r^2} = \frac{(1-\eta)\, W d^2}{16 r^2} \doteq 91.7\,\mathrm{W}\,.$$

where $1-\eta$ is the absorption coefficient.

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Problem DC ... lumberjack physicist

There is an idealised homogeneous tree with uniform, but negligible thickness and height h. We cut the tree next to the ground in such a way that it starts falling around a fixed point at the bottom. What is the velocity of the top of the tree immediately before it hits the ground?

Lego was trying to make a pen stand on its tip.

The potential energy of the tree is converted to rotational (kinetic) energy:

$$mg\Delta h = \frac{1}{2} J\omega^2$$

The tree is homogeneous, so its center of mass is at its midpoint

$$\Delta h = \frac{1}{2} h\,.$$

The moment of inertia of a rod rotating around an axis passing through its center of mass is

$$J_t = \frac{1}{12} m l^2\,,$$

and from Steiner's theorem, we find that when rotating around an axis passing through its endpoint, the moment of inertia is

$$J = J_t + m\left(\frac{h}{2}\right)^2 = \frac{1}{3} m h^2\,.$$

Finally, we need to realise that the velocity of the top of the tree is $v = \omega h$. Putting everything together, we get

$$v = \sqrt{3gh}\,.$$

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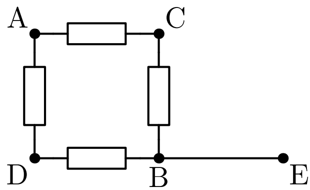
Problem DD ... black box

We have a black box with five terminals labelled A to E. We know that there are only 1 Ω resistors and wires inside. The table below shows values of resistance measured by a multimeter connected between all pairs of terminals. Draw an electrical diagram of the black box.

Hint *We did not use more than seven resistors.*

	A	B	C	D	E
A	0	1	3/4	3/4	1
B	1	0	3/4	3/4	0
C	3/4	3/4	0	1	3/4
D	3/4	3/4	1	0	3/4
E	1	0	3/4	3/4	0

First, note that B and E are connected by a wire. Therefore, the problem reduces to a black box with 4 terminals (A to D).



Next, we may notice symmetry between these four terminals. For each terminal, there is one "sister" terminal and the resistance between them is 1 Ω, while the resistance between this terminal and either of the remaining two is 3/4 Ω. Assuming we have ≤ 7 resistors, there are only two possible ways of connecting them symmetrically to all four terminals in such a way that all resistances are non-zero. There may either be a cycle of four resistors or the same cycle with pairs of opposite vertices connected by resistors (6 resistors in total). It's easy to see that only the first configuration fulfills the given conditions.

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Problem DE ... two cuboids

One upon a time, there were two cuboids with base areas S_1 and S_2, where $S_1 < S_2$. One day, the first one (the one with a smaller base) fell into the other one, since it was hollow. However, the first cuboid wasn't hollow, it had a height h and density ϱ_1. Therefore, it used that moment to think: what is the volume of a liquid with density $\varrho > \varrho_1$ which needs to be poured inside the second cuboid so that the first one stops sticking to its bottom?

Legolas was picking up a glass from a sink.

For the cuboid to stop sticking to the bottom of the other one, the buoyant force acting on it needs to compensate for the gravitational force

$$F_{\mathrm{vz}} = F_{\mathrm{g}}\,,$$
$$V_{\mathrm{p}}\varrho g = S_1 h \varrho_1 g\,,$$
$$V_{\mathrm{p}} = S_1 h \frac{\varrho_1}{\varrho}\,.$$

We know what part of the cuboid is submerged, and from this, we know that water reaches to a height

$$v=\frac{V_\mathrm{p}}{S_1}=h\frac{\varrho_1}{\varrho}\,.$$

Since water fills the space between the cuboids, its volume must be at least

$$V=v\Delta S=h\frac{\varrho_1}{\varrho}\left(S_2-S_1\right)\,.$$

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Problem DF ... fast crash

A particle with rest mass $m_1=2m_0$ is flying in the positive direction of the x-axis, with velocity $v_1=\frac{3}{5}c$. A particle with rest mass $m_2=3m_0$ is flying in the opposite direction with velocity $v_2=-\frac{4}{5}c$. These two particles collide (the collision is inelastic), creating a particle with rest mass m_3 flying with velocity v_3. Find its mass and velocity (including the direction).

Danka was reminiscing about a special relativity course.

In the collision, effects of special theory of relativity are important. We can write the law of conservation of relativistic mass (or energy) of the system

$$\frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}}+\frac{m_2}{\sqrt{1-\frac{v_2^2}{c^2}}}=\frac{m_3}{\sqrt{1-\frac{v_3^2}{c^2}}}\,,$$

and also the relativistic momentum conservation principle for the whole system

$$\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}}+\frac{m_2v_2}{\sqrt{1-\frac{v_2^2}{c^2}}}=\frac{m_3v_3}{\sqrt{1-\frac{v_3^2}{c^2}}}\,.$$

We express m_3 from the first equation, substitute it into the second equation and then express v_3 as

$$v_3=\frac{\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}}+\frac{m_2v_2}{\sqrt{1-\frac{v_2^2}{c^2}}}}{\frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}}+\frac{m_2}{\sqrt{1-\frac{v_2^2}{c^2}}}}\,.$$

After simplification, we get

$$v_3=\frac{m_1v_1\sqrt{c^2-v_2^2}+m_2v_2\sqrt{c^2-v_1^2}}{m_1\sqrt{c^2-v_2^2}+m_2\sqrt{c^2-v_1^2}}\,.$$

We plug in the numerical values and get $v_3=-\frac{1}{3}c$. The minus sign means that the particle will be flying in the negative direction of the x-axis.

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Problem DG ... air goes through

A dormitory room has temperature $t_{\mathrm{in}} = 15\,^\circ\mathrm{C}$. Outside, it is freezing, with temperature $t_{\mathrm{out}} = -5\,^\circ\mathrm{C}$. In the room, there are two people, each of them generating power $P_0 = 200\,\mathrm{W}$, a radiator with power $P_1 = 1500\,\mathrm{W}$ and a leaking simple glass window with area $S = 5\,\mathrm{m}^2$ and heat transfer coefficient $\Lambda = 0.73\,\mathrm{W{\cdot}m^{-2}{\cdot}K^{-1}}$. Find the volume V of cold air which flows into the room in one minute. Assume that this air is exchanged for an equivalent volume of "hot" air from the room. The room is perfectly thermally insulated otherwise. *Dodo almost froze.*

The heat flowing through the window is

$$P_2 = \Lambda S\ \Delta t\,,$$

where Δt is the difference between inside and outside temperatures. The power P which flows out of the room due to air flow outside is given by $P + P_2 = 2P_0 + P_1$. The air flow is then given by

$$P = c\dot{m}\Delta t = c\varrho Q_V \Delta t\,,$$

where $c = 1.0\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$ is the specific heat capacity of air at constant pressure, $\varrho = 1.28\,\mathrm{kg{\cdot}m^{-3}}$ is the density of air and Q_V is the volumetric flow of air through the imperfectly insulating window. The volume of air which flows out of the room during a time τ can be computed as $V = Q_V\tau$. In total, during $\tau = 60\,\mathrm{s}$, the volume of air which flows out of the room is

$$V = \frac{2P_0 + P_1 - \Lambda S\ \Delta t}{c\varrho\Delta t}\tau \doteq 4.3\,\mathrm{m}^3\,.$$

In one minute, approximately $4.3\,\mathrm{m}^3$ of cold air flows into the room.

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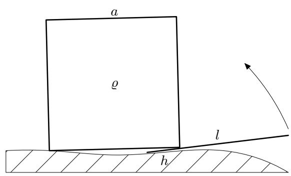
Problem DH ... lifting by lever

Find the minimum force F necessary to lift a steel cube with edge length $a = 1.5\,\mathrm{m}$ using a crowbar with length $l = 2\,\mathrm{m}$. The crowbar is a straight rod and the part of the crowbar that is inserted under the cube (in the middle of one of the bottom edges of the cube, perpendicularly to this edge) has length $h = 0.1\,\mathrm{m}$. The cube, the crowbar and the floor are rigid.
Dodo likes to save his effort.



From equality of torques at the point where the crowbar touches the cube (the middle of an edge), we can find a formula relating the force F which we're exerting on the crowbar and the force f with which the crowbar is acting on the edge of the cube

$$Fl = fh\,.$$

We're lifting the cube by rotating it around the floor edge opposite to the edge which touches the crowbar. When the torques due to gravity F_g and the force f cancel out, we get

$$F_g\frac{a}{2} = fa\,.$$

The force due to gravity can be expressed using the volume and density of the cube

$$f = \frac{1}{2}\varrho a^3 g\,,$$

and substituting for the force F, we get

$$F = \frac{1}{2}\varrho a^3 g\frac{h}{l} = 6.5\,\mathrm{kN}\,,$$

so the minimum required force we need to exert on the crowbar is $6.5\,\mathrm{kN}$.

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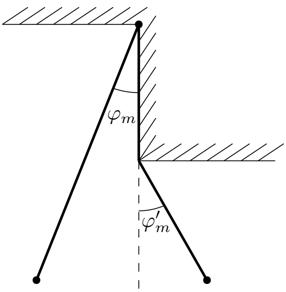
Problem EA ... asymmetrical pendulum amplitude



There is a pendulum, depicted schematically (with distances and angles not corresponding to this problem) in the figure. When swinging on the left side, the pendulum has length l and angular amplitude φ_m. On the other side, half of the rope is stopped by a wall and the remaining half swings with an angular amplitude φ'_m. If $\varphi_m = 5.0\,^\circ$, what is φ'_m? Assume that it is an ideal mathematical pendulum without energy loss.

Karel likes asymmetrical oscillations.

Since we're implicitly assuming that a pendulum is located in uniform gravity, we know that the point mass at the end of the rope reaches the same height in each extreme point. Let's denote this height by h. Now, we can use two right triangles. From the larger one, we get

$$\cos\varphi_m = \frac{l-h}{l} = 1 - \frac{h}{l}$$

and from the smaller one,

$$\cos\varphi'_m = \frac{\frac{l}{2}-h}{\frac{l}{2}} = 1 - \frac{2h}{l}\,.$$

Expressing h from each equation and putting it together, we get

$$\cos\varphi'_m = 2\cos\varphi_m - 1\,, \qquad \Rightarrow \qquad \varphi'_m = \arccos\left(2\cos\varphi_m - 1\right) \doteq 7.07\,^\circ\,.$$

The amplitude on the other side is $7.07\,^\circ$.

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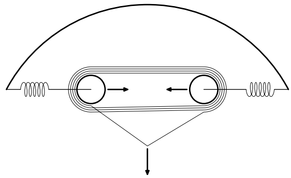
Problem EB ... Jachym's bow



Jachym's new compound bow can be approximated as two pulleys with negligible diameters separated by a distance y_0 with a string wound around them in four loops. The string is a closed loop with a constant length $8y_0$. The pulleys can only move along the line passing through their centres and each pulley is connected to the rigid frame of the bow by a spring with a spring constant k. Jachym now pulls one of the threads of the string back, perpendicularly to the line passing through the pulleys. Determine how the force he must exert on the string depends on the drawing distance x. At the beginning, the tension in the springs is zero.

The original version had a longbow. Also, the author was different.

Let's denote the distance between the pulleys by y. We know that initially, $y = y_0$. The total length of the string is $l = 8y_0$. When one of the threads is pulled back by x, the length of this part of the string changes to

$$z = 2\sqrt{x^2 + \left(\frac{y}{2}\right)^2} = \sqrt{4x^2 + y^2}\,.$$

The remaining seven threads remain straight between the pulleys, so the total length of the string is $l = 7y + z$. This length has to remain constant, and from this, we get

$$\begin{aligned} 8y_0 &= 7y + z\,, \\ 0 &= 12y^2 - 28y_0 y + 16y_0^2 - x^2\,, \\ y &= \frac{7y_0 \pm \sqrt{y_0^2 + 3x^2}}{6}\,. \end{aligned}$$

We require $y \leq y_0$, so only the solution with the minus sign is relevant. Each spring exerts a force

$$F_y = \frac{1}{2}\left(y_0 - y\right)k\,.$$

When Jachym exerts a force F_x along a distance $\mathrm{d}x$, the work done is $\mathrm{d}W = F_x \mathrm{d}x$. This is used to change the distance of the pulleys by $\mathrm{d}y$ and therefore to extend the springs by $-\mathrm{d}y$. The change in the energy stored in each spring is

$$\mathrm{d}W_\mathrm{p} = -\frac{1}{2}F_y \mathrm{d}y\,.$$

The total change in the energy of both springs is $\mathrm{d}W = 2\mathrm{d}W_\mathrm{p} = -F_y\mathrm{d}y$. From this, we get

$$F_x = -\frac{\mathrm{d}y}{\mathrm{d}x}F_y = -\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\left(y_0 - y\right)k\,.$$

Now we only need to differentiate y by x, plug this back in and after some manipulation, we get

$$F_x = \frac{1}{24}kx\left(\frac{y_0}{\sqrt{y_0^2 + 3x^2}} - 1\right)\,.$$

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Problem EC ... laser

A teacher is illuminating a blackboard with a green laser pointer. The cross-sectional area of the ray is $1 cm^2$ *and the power of the laser is* $1 mW$*. How many photons are located in one metre of the ray? The wavelength of green light is* $530 nm$*.*

Matěj got the idea during a lecture on optics.

Let's denote the time it takes light to travel the distance $l = 1\,\mathrm{m}$ by t. During this time, the laser radiates an energy Pt, where P is its power. This energy must be equal to the energy of all photons NE, where N is the number of photons in this distance and $E = hf = \frac{hc}{\lambda}$ is the energy of one photon. Here, h is Planck's constant, f is the frequency of a photon, λ is its wavelength and c is the speed of light (clearly $t = l/c$). Finally, we obtain

$$Pt = NE\,,$$

$$P\frac{l}{c} = N\frac{hc}{\lambda}\,,$$

$$N = \frac{P\lambda l}{hc^2} \approx 8{,}89 \cdot 10^6\,.$$

In one metre, there are almost nine million photons.

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Problem ED ... 1D billiards

Consider an infinite trough and $N = 17$ *identical balls which can only move inside the trough. The positions and velocities of the balls may be chosen arbitrarily. What is the maximum possible number of collisions that can occur? All collisions are perfectly elastic and friction inside the trough is negligible.* *Matúš was watching a billiards tournament.*

Let's take a graph of time-dependent positions of all balls. The movement of each ball forms a line, since they move with uniform velocities. Collisions are represented by intersections of these lines and since the balls are identical, they just swap their velocities during an elastic collision. This means that one ball keeps moving along the trajectory of the other and vice versa, so the lines don't change slopes. The maximum possible number of collisions is the maximum number of intersection points of N lines, which is

$$p = \frac{N\,(N-1)}{2}\,.$$

In our case, $N = 17$, so at most 136 collisions are possible.

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Problem EE ... wailing vibes

Suppose that you look at a particle (a point mass) oscillating at the end of a spring at a random moment in time. What is the probability that the distance of this point from the equilibrium position does not exceed $y_1 = 1.0\,\mathrm{cm}$*? The amplitude of the particle's oscillations is* $y_\mathrm{m} = 3.0\,\mathrm{cm}$ *and there is no damping force.* *Karel was staring at springs.*

The system we're considering is a linear harmonic oscillator and the position of the point mass can be described as $y(t) = y_\mathrm{m} \sin(\omega t + \varphi_0)$. The phase constant φ_0 has no effect on the probability we are interested in, so we can assume $\varphi_0 = 0$. To further simplify the expression above, let's define $\alpha = \omega t$.

As for the movement itself, now described by $y(t) = y_\mathrm{m} \sin\alpha$, it is 2π-periodic. Its absolute value, though, is π-periodic and of greater interest to us, since we consider only the distance from the equilibrium position. Furthermore, this absolute value is axisymmetric with respect to $x = \pi/2$. This means that we can consider just the interval $[0, \pi/2]$. The total time corresponding to this interval is proportional to the total angle $\pi/2$. The time during which the position of the particle is below y_1 corresponds to the part between time/angle zero and the point where $y_1 = y_\mathrm{m} \sin\alpha$, or $\sin\alpha = 1/3 \quad\Rightarrow\quad \alpha = 0.340$. The probability of the distance from equlibrium being smaller than y_1 is then $P = \alpha/(\pi/2) \doteq 21.6\,\%$. It is more likely for the particle to be at a point more distant from the equilibrium position than the given distance.

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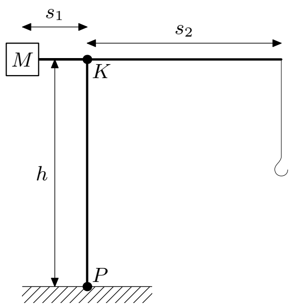
Problem EF ... crane driver

A crane driver forgot how much weight his crane can lift up. The height of the crane is $h = 80\,\mathrm{m}$ and a counterweight with mass $M = 10\,\mathrm{t}$ is positioned at a distance $s_1 = 10\,\mathrm{m}$ from the cabin (point K). A hook (ignore its weight) is hanging freely $s_2 = 60\,\mathrm{m}$ from the cabin. The linear density of the whole construction is $\lambda = 10\,\mathrm{kg{\cdot}m^{-1}}$. The crane driver knows that the weakest points of the crane are its foot (point P), where it is anchored to the ground, and its joint (point K). He does not remember the critical magnitude of torque which can act at this point (in either direction), but he knows that after exceeding this value, the crane would collapse. How much weight can the crane driver certainly lift up?



Matěj's dream job.

When there is no additional weight on the crane, a torque τ is acting on the foot. After lifting up a sufficient weight, the final torque starts acting in the other direction (and therefore pulls the crane to the right). Everything is ok while the magnitude of this new torque doesn't exceed τ. Obviously, there is no difference between the torques at the points K and P.

The mass of the whole arm of the crane is $\lambda(s_1 + s_2)$ and its center of gravity is lying $(s_2 - s_1)/2$ to the right from the point K. The arm acts with a torque $-\lambda(s_1 + s_2)(s_2 - s_1)/2$ (against the torque of the weight, Ms_1). We have

$$\tau = Ms_1 - \frac{\lambda(s_1 + s_2)(S_2 - s_1)}{2}\,.$$

The resulting torque is $-\tau$ when the weight itself causes torque 2τ. Let m be the maximum mass of the weight.

$$ms_2 = 2\tau\,,$$
$$m = \frac{2Ms_1}{s_2} - \frac{\lambda}{s_2}\left(s_2^2 - s_1^2\right) = 2{,}750\,\mathrm{kg}\,.$$

Thus the maximum allowed mass of the weight is 2,750 kg.

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Problem EG ... Christmas snowman reloaded

Consider a large snowball with radius R and acceleration due to gravity at its surface equal to g. Let's use this snowball as the base of a snowman, which is made up of an infinite number of snowballs with the same density, whose radii form an infinite geometric sequence with a coefficient (common ratio, quotient) 1/2. Adjacent snowballs always touch and the centres of all snowballs are collinear. Compute the intensity of gravitational field at the top of this snowman. The snowman is not located in any external gravitational field.

Matúš wanted to build a really large snowman.

From Newton's gravitational law, we know that intensity of gravitational field is proportional to the first power of length, so if all lengths are halved, the intensity is also halved. This halved snowman is identical to the original snowman without the first snowball, which gives us an equation

$$g' = g_1 + \frac{g'}{2}\,;$$

g_1 is the intensity due to the first snowball. Since the radii of snowballs form a geometric sequence, the height of the snowman is

$$h = \frac{2R}{1-\frac{1}{2}} = 4R\,.$$

The distance of the center of the first snowball from the top is $3R$, so the intensity due to this snowball at the top is nine times smaller than at the surface of this snowball, and substituting in the formula above, we get

$$g' = \frac{g}{9} + \frac{g'}{2} = \frac{2}{9}g\,.$$

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Problem EH ... lunar mining

Once upon a time mankind finally managed to build a lunar base. Instead of scientific research, however, they started plundering the Moon when they discovered an invaluable material named Fykosium, with density $\varrho = 5{,}000\,\mathrm{kg\cdot m^{-3}}$, and started mining it homogeneously from the surface of the Moon. The mining operations were so intensive that the radius of the Moon decreased and it wasn't possible to watch a full solar eclipse from the surface of Earth anymore. Calculate the mass of Fykosium that was mined out. The semi-major axis of the Moon's orbit is $a = 384{,}400\,\mathrm{km}$ and its eccentricity is $e = 0,0549$. Assume that the orbit of the Earth around the Sun is a circle with radius $r = 1.496\cdot 10^{11}\,\mathrm{m}$. The Sun has radius $R_\mathrm{S} = 6.96\cdot 10^{8}\,\mathrm{m}$, the equatorial radius of Earth is $R_\mathrm{Z} = 6{,}378\,\mathrm{km}$, the radius of the Moon is $R_\mathrm{M} = 1{,}738\,\mathrm{km}$. Assume that $R_\mathrm{Z} \ll r$ and that all the Fykosium that was mined out remains on an orbit around the Moon, so the orbit of the Moon around the Earth has not changed.

Jáchym and Jirka were thinking about the future of mankind.

From similarity of triangles, we get

$$\frac{R_\mathrm{S}}{r} = \frac{R}{a(1-e) - R_\mathrm{Z}}\,,$$

where R is the radius of the Moon after Fykosium is mined out. Therefore,

$$R = \frac{R_\mathrm{S}}{r}\left(a(1-e) - R_\mathrm{Z}\right) \doteq 1660.5\,\mathrm{km}\,.$$

The volume V and mass m that was mined out can then be computed as

$$V = \frac{4}{3}\pi(R_\mathrm{M}^3 - R^3)\,,$$
$$m = V\varrho = \frac{4}{3}\pi\varrho(R_\mathrm{M}^3 - R^3) \doteq 1.41 \cdot 10^{22}\,\mathrm{kg}\,.$$

The mass of Fykosium that was mined out is approximately $1.41 \cdot 10^{22}$ kg.

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Problem FA ... Christmas snowman

Matúš built a snowman in the following way: He made a snowball with radius R. Then, he made a second snowball with radius $R/2$ and placed it on top of the first snowball. On top of the second snowball, he put a third one with radius $R/4$, and so on, placing an infinite number of balls on top of each other, always with a halved radius. All snowballs have the same density. At what height above the ground is the centre of mass of the resulting snowman?

Matúš wanted to build a snowman.

Let's compute the position of the centre of mass as a weighted average of positions of centres of mass of individual balls, where the weights are masses of the balls, i.e. the position of the centre of mass x_T satisfies

$$Mx_T = \sum_{i=1}^{\infty} m_i x_i\,,$$

where M is the mass of the snowman. The origin of the coordinate system can be chosen arbitrarily, so let's choose it in the centre of the first ball, so that the first term in the sum would be zero. The rest of the sum can be rewritten as

$$Mx_T = m_1 \cdot 0 + \sum_{i=2}^{\infty} m_i x_i = M' x'_T\,,$$

where M' and x'_T are the mass and position of the center of mass of the snowman without the first ball, which is identical to the original snowman, but with halved dimensions. Since mass is proportional to the cube of radius and the position of the centre of mass is proportional to radius, we get $M' = M/8$ and $x'_T = x_T/2 + 3R/2$ ($x_T/2$ is the distance from the centre of the second ball). Substituting for M' and x'_T, we get

$$Mx_T = \frac{M}{8} \cdot \frac{3R + x_T}{2}\,,$$
$$x_T = \frac{R}{5}\,,$$

but x_T is the distance of the centre of mass from the centre of the first ball, which has radius R, so the height of the centre of mass above the ground is

$$h_T = \frac{6R}{5}\,.$$

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Problem FB ... slide farther

You are standing on a horizontal plane and throwing an object from zero height. What is the optimal angle of your throw (with respect to the horizontal plane) if you want the object to come to rest after sliding as far as possible? The friction coefficient between the thrown object and the ground is $f = \sqrt{3}/4$, the collision of the thrown object with the ground is perfectly inelastic and it does not rotate.

Matěj was imagining throwing snowballs, since there was no snow :,(.

At first, the object is flying on a parabolic trajectory. Then, it collides with the ground and loses the vertical component of its velocity; the horizontal component remains the same. Then, its velocity decreases due to friction until it comes to rest. Let's assume that we're throwing with a velocity v under an angle φ with respect to the horizontal plane. A well-known formula for the motion of a projectile gives us the distance at which it collides with the ground

$$s_1 = \frac{2v^2}{g}\sin\varphi\cos\varphi\,.$$

After the collision, the object is moving horizontally with velocity $v_x = v\cos\varphi$. There is a uniform frictional deceleration $a_\mathrm{t} = fg$ acting on it. The time of sliding is

$$t = \frac{v_x}{a_\mathrm{t}} = \frac{v_x}{fg}\,.$$

During this time, the object crosses a distance

$$s_2 = v_x t - \frac{1}{2}a_\mathrm{t}t^2 = \frac{v_x^2}{2fg} = \frac{v^2}{2fg}\cos^2\varphi\,.$$

It comes to rest at a distance

$$s = s_1 + s_2 = \frac{2v^2}{g}\sin\varphi\cos\varphi + \frac{v^2}{2fg}\cos^2\varphi\,.$$

We're looking for the maximum of the function $s(\varphi)$. Let's compute the first derivative and solve for φ which makes it zero

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\varphi} &= \frac{v^2}{g}\left(2\cos^2\varphi - 2\sin^2\varphi - \frac{1}{f}\cos\varphi\sin\varphi\right)\,,\\ 0 &= 2f\left(\cos^2\varphi - \sin^2\varphi\right) - \cos\varphi\sin\varphi\,,\\ 0 &= 2f\cos 2\varphi - \frac{1}{2}\sin 2\varphi\,,\\ \varphi &= \frac{1}{2}\arctan 4f\,.\end{aligned}$$

The total distance s is minimum at the endpoints of the interval of φ we consider, i.e. for $\varphi = 0\,^\circ$ and $\varphi = 90\,^\circ$. The extremum we found therefore has to be the maximum we're looking for. Substituting $f = \sqrt{3}/4$, we get $\varphi = 30\,^\circ$.

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Problem FC ... bouncing pendulum

There is a pendulum formed by a steel ball (radius $r = 0.5\,\mathrm{cm}$, specific heat capacity $c = 452\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$, density $\varrho = 7{,}850\,\mathrm{kg{\cdot}m^{-3}}$) attached to a wall by a rope with length $l = 1.0\,\mathrm{m}$, so the ball always bounces off the wall in the bottommost point of its trajectory. During a bounce, the ball loses energy and heats up. The velocity with which the ball bounces off is given by the coefficient of restitution $k = v_1/v_0 = 0.90$ (v_0 and v_1 are the velocities before and after the bounce, respectively). We release the pendulum from an initial position given by a displacement angle $\alpha = 45^\circ$. How much higher is the temperature of the ball after three bounces? Assume that half of the energy lost during bounces is used to heat the ball.

Karel was bashing his head against a wall.

It turns out that the size and mass of the ball aren't relevant parameters, but we'll calculate and use it anyway for clarity.

First, let's compute the initial mechanical energy of the pendulum. This energy corresponds to the potential energy of the pendulum in the initial position, which later transfers into kinetic energy and starts getting dissipated. The initial energy is

$$E_0 = mgl\,(1 - \sin\alpha) \doteq 0.0118\,\mathrm{J}\,,$$

where we substituted for the mass $m = \varrho V = 4\pi\varrho r^3/3 \doteq 4.11\,\mathrm{g}$.

The restitution coefficient is defined as a ratio of velocities. Since kinetic energy is proportional to squared velocity[1] the ratio of energy after the collision to energy before it is

$$K = \frac{E_1}{E_0} = \frac{v_1^2}{v_0^2} = k^2\,.$$

This way, we find out that the mechanical energy of the pendulum after two bounces is $E_2 = KE_1 = k^2 E_1 = k^4 E_0$ and after three bounces, it's $E_3 = k^6 E_0$. The total mechanical energy dissipated into other forms of energy (heat, sound, etc.) is

$$\Delta E = E_0 - E_3 = \left(1 - k^6\right) E_0 = mgl\,(1 - \sin\alpha)\left(1 - k^6\right) \doteq 5.53\,\mathrm{mJ}\,.$$

Half of this energy, $\Delta E/2 = Q \doteq 2.77\,\mathrm{mJ}$, is spent on heating the ball. The temperature increase can be found using the formula $Q = mc\Delta T$. Substituting for the already computed heat Q, we get

$$\frac{\Delta E}{2} = mc\Delta T \qquad \Rightarrow \qquad \Delta T = \frac{\left(1 - k^6\right) E_0}{2mc} = \frac{1 - k^6}{2}\frac{gl}{c}\,(1 - \sin\alpha) \doteq 1.5\,\mathrm{mK}\,.$$

The ball heats up by a minuscule 1.5 mK.

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[1] Indeed if we considered rotational energy, the result is still proportional to squared velocity, because the rotational energy is proportional to the squared angular velocity which is proportional to the velocity in this case.

Problem FD ... capacitors

Initially, there is an LC circuit composed of one coil and one parallel-plate capacitor. The distance between the plates of the capacitor is $d_0 = 4.0\,\mathrm{mm}$*, the surface area is* $S = 500\,\mathrm{cm}^2$ *and the inductance of the coil is* $L = 20\,\mathrm{mH}$*. After some time, we insert a conducting plate with thickness* $r = 0.5\,\mathrm{mm}$ *inside the capacitor exactly in the middle between its plates. How does the resonance frequency of the circuit change? Express the ratio of new to old frequency.*

The initial capacitance of the capacitor is

$$C_0 = \frac{\varepsilon S}{d_0}\,,$$

where ε is the permittivity of the medium inside it. If we insert a conducting plate in the middle, we effectively split it into two smaller capacitors. The distance between plates in each of them is

$$d = \frac{d_0 - r}{2}\,,$$

so each of them has capacitance

$$C' = \frac{2\varepsilon S}{d_0 - r}\,.$$

The new capacitors are connected in series. Their capacitances combine in the same way as resistances of parallel resistors, so the resulting capacitance is

$$C = \frac{C'^2}{C' + C'} = \frac{1}{2}C' = \frac{\varepsilon S}{d_0 - r}\,.$$

In the original LC circuit, the current flowing through both elements is identical, while the voltages on them are opposite to each other. This leads to a differential equation

$$\ddot{I} + \frac{1}{LC_0}I = 0\,.$$

Its solutions are some complex exponentials, in general

$$I = A\mathrm{e}^{\mathrm{i}\omega_0 t} + B\mathrm{e}^{-\mathrm{i}\omega_0 t}\,,$$

where A, B are constants and ω_0 is the angular frequency

$$\omega_0 = \frac{1}{\sqrt{LC_0}}\,.$$

The situation after the plate is inserted gives the same equations, we only need to use the capacitance C computed above instead of C_0. The resulting ratio of frequencies is

$$\frac{f}{f_0} = \frac{\omega}{\omega_0} = \sqrt{\frac{C_0}{C}} = \sqrt{1 - \frac{r}{d_0}} = \sqrt{\frac{7}{8}} \doteq 0.94\,.$$

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Problem FE ... Christmas snowman reloaded reloaded

Consider a large snowball with radius R. Let's use this snowball as the base of a snowman, which is made up of an infinite number of snowballs with the same density, whose radii form an infinite geometric sequence with a coefficient (common ratio, quotient) $1/2$. Adjacent snowballs always touch and the centres of all snowballs are collinear. Compute the moment of inertia of the snowman with respect to an axis passing through the top of the snowman and perpendicular to the line passing through the centres of all snowballs. The total mass of the whole snowman is M. *Matúš really really likes snowmen.*

The moment of inertia will be in the form $I = kMR^2$, where k is a dimensionless constant. The mass of the snowman is proportional to the cube of R, so the moment of inertia is proportional to R^5. Let's split the moment of inertia into two parts: the moment of inertia of the largest snowball I_o and the moment of inertia of the remaining snowballs I_z. The resulting moment of inertia is their sum

$$I = I_o + I_z\,.$$

The moment of inertia of the largest snowball can be computed using Steiner's theorem

$$I_o = \frac{2}{5}mR^2 + md^2$$

where d is the distance of center of the largest snowball from the top and m is its mass.

The height of the snowman is the sum of the diameter of the largest snowball and the height of the rest of the snowman. You may notice that the rest of the snowman is identical to the original snowman with all dimensions halved, so its height is also half of the original snowman's height. The diameter of the largest ball is $2R$, which gives us the height of the snowman $4R$ and the distance of the centre of the largest snowball from the top $d = 3R$.

Since the dimensions of the remaining part of the snowman are halved compared to the original and the moment of inertia is proportional to R^5, the moment of inertia of the rest of the snowman is 32 times smaller than the moment of inertia of the original snowman. Since mass is proportional to R^3, the mass of the rest is $1/8$ of the whole snowman's mass, so the mass of the largest snowball is $m = 7/8M$. Substituting everything in the first equation, we get

$$\begin{aligned} I &= \frac{2}{5}\cdot\frac{7}{8}MR^2 + \frac{7}{8}M\left(3R\right)^2 + \frac{I}{32}\,,\\ \frac{31}{32}I &= \frac{329}{40}MR^2\,,\\ I &= \frac{1\,316}{155}MR^2\,. \end{aligned}$$

We may notice that the largest snowball contributes the most by far to the moment of inertia - if we assumed that the whole mass of the snowman is at its center, we'd get $I \approx 9MR^2$, while $k \doteq 8.49$.

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Problem FF ... analytical bouncing ball

Vašek likes playing with bouncing balls; specifically, he's interested in the restitution coefficient. He decided to test a bouncing ball on a larger scale, so he threw it from the 15-th floor of the *dorm, at a height $h_1 = 50\,\mathrm{m}$ above the ground, under a randomly chosen angle. At the 20-th floor (height $h_2 = 65\,\mathrm{m}$), however, there is the dorm supervisor, who most definitely would not like to see someone throwing things from windows. Since the dorm supervisor does not lean out of the window, he only sees outside under the angle $\alpha = 60^\circ$ downwards with respect to the horizontal plane. What is the maximum velocity with which Vašek can throw the bouncing ball so that it is guaranteed that the supervisor cannot see it at any point of its trajectory (before it impacts the ground)?*

Consider a coordinate system such that the y-axis corresponds to the wall and the x-axis corresponds to the ground. The supervisor's line of sight is given by the formula $y = h_2 - x\tan\alpha$. Next, we need to determine the trajectory of the ball depending on the initial velocity v and the angle φ under which it's thrown. Let's define the angle φ as the angle between the initial velocity vector and the horizontal plane in the same way as α (positive when throwing down). Then, the ball follows a regular ballistic trajectory with coordinates at time t after the throw given by

$$x = tv\cos\varphi\,,$$
$$y = h_1 - tv\sin\varphi - \frac{1}{2}gt^2\,.$$

We don't need the time dependence at all, only the relation between y and x, so let's express t from the first formula and plug it in the second formula

$$y = h_1 - x\tan\varphi - \frac{gx^2}{2v^2\cos^2\varphi}\,.$$

Now we can find the intersection of the bouncing ball's trajectory and the dorm supervisor's line of sight. Mathematically, we're looking for x such that both curves' y are equal

$$h_2 - x\tan\alpha = h_1 - x\tan\varphi - \frac{gx^2}{2v^2\cos^2\varphi}\,,$$
$$0 = x^2\frac{g}{2v^2\cos^2\varphi} + x\left(\tan\varphi - \tan\alpha\right) + h_2 - h_1\,.$$

This is a quadratic equation. If it has a positive discriminant, it means that the bouncing ball crosses the supervisor's line of sight twice, which we want to avoid. The critical case is when the discriminant is 0, which is when the ball only touches the line of sight.

Let's denote the velocity which leads to discriminant 0 for a fixed φ by $v_0(\varphi)$. We obtain an equation

$$\left(\tan\varphi - \tan\alpha\right)^2 - 4\frac{g}{2v_0(\varphi)^2\cos^2\varphi}\left(h_2 - h_1\right) = 0\,.$$

Next, let's express

$$v_0(\varphi)^2 = \frac{2g\left(h_2 - h_1\right)}{\cos^2\varphi\left(\tan\varphi - \tan\alpha\right)^2}\,. \tag{1}$$

We're supposed to find the maximum v such that for each φ, the ball isn't seen by the supervisor. This velocity is the minimum of $v_0(\varphi)$.

The right hand side of (1) is always non-negative, so if we want to minimise $v_0(\varphi)$, we only need to minimise its square. In addition, the numerator doesn't depend on φ, so we only need to maximise the denominator. If we require its derivative with respect to φ to be 0, we get

$$\left(\tan\varphi - \tan\alpha\right)\left(\left(\tan\varphi - \tan\alpha\right)\cos\varphi\sin\varphi - 1\right) = 0\,.$$

One solution is clearly $\varphi = \alpha$, which means the denominator in (1) is 0. Its physical meaning is throwing in parallel with the dorm supervisor's line of sight (or more vertically), in which case any velocity is allowed. However, this case isn't interesting for us, so the second parenthesised expression must be zero. Further simplifications give us a condition

$$\cos\varphi\left(\tan\alpha\sin\varphi + \cos\varphi\right) = 0\,.$$

For $\cos\varphi = 0$, we get two possible angles: $90\,^\circ$ (vertically down), which again isn't interesting, and -90° (vertically up). It's easy to compute that $v_0(-90^\circ) = \sqrt{2g\left(h_2 - h_1\right)}$.

The condition $\tan\alpha\sin\varphi + \cos\varphi = 0$ can be again simplified to

$$\tan\varphi = -\cot\alpha\,,$$

which gives $\varphi = -30\,^\circ$ (throwing upwards under an angle $30\,^\circ$). If we plug this angle into (1), we get

$$v_0(-30^\circ) = \sqrt{\frac{2g\left(h_2 - h_1\right)}{\tan^2\alpha + 1}} = \sqrt{\frac{1}{2}g\left(h_2 - h_1\right)}\,.$$

This is less than $v_0(-90^\circ)$, so it's the velocity we were looking for.

For the given numerical values, we get $v_0(-30^\circ) = 8.58\,\mathrm{m\cdot s^{-1}}$.

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Problem FG ... ice hockey sticks

Jirka and Jachym wanted to compete in ice hockey puck shooting. Both of them were shooting from the middle of a rink, at a distance $l = 19.50\,\mathrm{m}$ from a hockey goal. Jirka decided to shoot the puck along the surface of the ice into the middle of the goal with velocity $v_0 = 20.00\,\mathrm{m\cdot s^{-1}}$, while Jachym was shooting into the left gibbet (= the top left edge of the goal) with velocity u. By total accident, both of them scored a goal at the same time t from shooting. Calculate the velocity u of Jachym's shot. The dimensions of the hockey goal are $1.830\,\mathrm{m} \times 1.220\,\mathrm{m}$ (width $\times$ height), the friction coefficient of a puck on ice is $f = 0.150$ and the mass of the puck is $m = 170.0\,\mathrm{g}$. Calculate the result in $\mathrm{m\cdot s^{-1}}$ to 4 significant figures.

Jirka was watching ice hockey.

The friction force acting on a hockey puck with mass m sliding on ice is $F_\mathrm{t} = fmg$. Therefore, it is moving with acceleration $a = -fg$ and if the initial velocity is v_0, we're solving an equation

$$l = v_0 t + \frac{1}{2}at^2 = v_0 t - \frac{1}{2}fgt^2\,.$$

We express the time from shooting the puck to striking a goal from the quadratic equation and get

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 2fgl}}{fg}\,.$$

When we substitute $l = 0$, the time is equal to zero only for the root with the minus sign, so we consider only

$$t = \frac{v_0 - \sqrt{v_0^2 - 2fgl}}{fg} \doteq 1.012{,}7\,\mathrm{s}\,.$$

The trajectory of the puck satisfies

$$x = u\ t \cos\alpha\,,$$
$$y = u\ t \sin\alpha - \frac{1}{2} g t^2\,,$$

where α is an angle between the vector of velocity and the horizontal plane, and y is the height of the hockey goal. The horizontal distance is

$$x = \sqrt{l^2 + \left(\frac{b}{2}\right)^2} \doteq 19.52\,\mathrm{m}\,,$$

where b is the width of the goal. After substituting for u from the first equation, we get

$$\tan\alpha = \frac{y + \frac{1}{2} g t^2}{x}\,,$$
$$\alpha \doteq 17.75\,^{\circ}\,.$$

Now we can express the velocity from the first equation

$$u = \frac{x}{t\cos\alpha} \doteq 20.24\,\mathrm{m{\cdot}s^{-1}}\,.$$

We may notice that the mass of the puck wasn't needed for solving the problem.

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Problem FH ... flat-Earth reloaded reloaded

It is well-known that the Earth is flat and supported by elephants. Above it, there is a dome from which the Sun, the Moon and stars are hanging. If the Moon with mass $7.348 \cdot 10^{22}$ kg was hanging from a steel rope with a (not necessarily uniform) circular cross-section, what would be the radius of its cross-section at the point of contact with the dome? The length of the rope is $l = 3{,}200\,\mathrm{m}$, the yield stress of steel is $700\,\mathrm{MPa}$ and its density is $7{,}850\,\mathrm{kg{\cdot}m^{-3}}$. The Moon is placed in uniform gravity g. Neglect all other gravitational effects.

Jáchym has a weakness for reloaded reloaded problems.

At the bottommost point, the force acting on the rope is just caused by gravity of the Moon mg, where m is the mass of the Moon. At this point, the radius of the rope is

$$r_0 = \sqrt{\frac{mg}{\pi\sigma}}\,,$$

where σ is the yield stress.

Let's denote the radius of the rope at distance x from this point by $r(x)$. If we move upwards by a small distance $\mathrm{d}x$, the mass of the rope underneath us increases by $\mathrm{d}m = \pi r^2 \varrho \mathrm{d}x$. If the radius increases by $\mathrm{d}r$, the cross-sectional area increases by $\mathrm{d}S = 2\pi r \mathrm{d}r$. Now we only need to realise that increasing the mass also increases the force which the rope needs to bear by $g\mathrm{d}m$, while increasing the cross-sectional area increases the force the rope can bear by $\sigma \mathrm{d}S$. These two expressions need to be equal, so we get

$$g\mathrm{d}m = \sigma \mathrm{d}S \;,$$
$$\frac{\varrho g}{2\sigma}\mathrm{d}x = \frac{\mathrm{d}r}{r}\,.$$

Integrating both sides, we obtain a solution

$$\int_0^x \frac{\varrho g}{2\sigma}\mathrm{d}x = \int_{r_0}^r \frac{\mathrm{d}r}{r}\,,$$
$$\frac{\varrho g}{2\sigma}x = \ln\left(\frac{r}{r_0}\right) = \ln\left(r\sqrt{\frac{\pi\sigma}{mg}}\right)\,.$$

Now it's easy to express

$$r(x) = \sqrt{\frac{mg}{\pi\sigma}}\exp\left(\frac{\varrho g}{2\sigma}x\right)\,.$$

We're interested in radius at height l, which is

$$r(l) = \sqrt{\frac{mg}{\pi\sigma}}\exp\left(\frac{\varrho g}{2\sigma}l\right) \doteq 2.16\cdot 10^7\,\mathrm{m}\,.$$

As expected, the radius we got is smaller than in the previous case, but it still doesn't make any sense.

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Problem GA ... centrifuge

We pour water into a hollow cylinder with base radius R to a height h_0. Then, we spin the cylinder up to a constant angular velocity. Determine the angular velocity at which the water level in the centre just touches the bottom of the cylinder.

Matúš wanted to dry the centre of a bucket.

The water level follows an equipotential surface created by the gravitational potential $\varphi_g = gh$ and the potential due to the centrifugal force, which can be determined by integrating the centrifugal acceleration over distance from the centre

$$\varphi_\mathrm{o} = -\int_0^r a_\mathrm{o}\mathrm{d}r = -\int_0^r \omega^2 r\mathrm{d}r = -\frac{1}{2}\omega^2 r^2\,,$$

where ω is the angular velocity. The sum of both potentials must be constant at the water surface, thus

$$gh - \frac{1}{2}\omega^2 r^2 = k\,.$$

We can use this to express the water level as a function of the distance from the centre r. Water is incompressible, so the volume of water in the container is constant. We can use this to determine the value of k. The initial volume was $V = \pi R^2 h_0$. We are interested in the case where $h \geq 0$ (with the equality for $r = 0$), so we only need to integrate

$$V = \int_0^R \mathrm{d}V = \int_0^R 2\pi r \mathrm{d}r h = \int_0^R \frac{2\pi r}{g}\left(k + \frac{1}{2}\omega^2 r^2\right)\mathrm{d}r = \frac{\pi R^2}{g}\left(k + \frac{1}{4}\omega^2 R^2\right) .$$

Comparing this to the original volume, we get

$$k = gh_0 - \frac{1}{4}\omega^2 R^2 ,$$

$$h = h_0 + \frac{\omega^2}{4g}\left(2r^2 - R^2\right) .$$

Using $r = 0$, $h(r = 0) = 0$, we easily find the required angular velocity

$$\omega = \frac{2}{R}\sqrt{gh_0} .$$

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Problem GB ... drinking tea 2.0

Consider a cylindrical container with height H and radius R, filled with tea with density ϱ. There is a little hole with area $s \ll S$ at the bottom of the container. This hole is used to attach a horizontal tube with a faucet and a manometer (a vertical tube, open at the top) to the container. Both tubes have radii r. The distance between the side of the container and the manometer is a, the distance between the manometer and faucet (the end of the horizontal tube) is b. Find the formula describing the height of liquid in the manometer $l(t)$ over time after the faucet is opened. Assume that this height adapts instantly to changes in flow in the horizontal tube. The tea is incompressible with dynamic viscosity η and surface tension σ. Assume laminar flow in the tube and $r \ll a, b \ll H, R$.
Hint In case of laminar flow, the pressure difference Δp between the ends of a tube satisfies Poiseuille's equation

$$\Delta p = \frac{8\eta l Q}{\pi R^4} ,$$

where R is the diameter of the tube, l its length and Q the volumetric flow rate.

Dodo chronically thinks in the canteen instead of eating.

From the Poiseuille equation, using the hydrostatic pressure at the bottom of the container $H\varrho g$ as the pressure difference between the start and the end of the tube, we get

$$H\varrho g = \frac{8\eta\left(a + b\right)Q}{\pi r^4} .$$

From the continuity equation, we know that this flow rate must correspond to a decrease in the tea level in the container.

$$Q = -\pi R^2 \dot{H} .$$

Plugging this into the previous equation, we get a simple separable differential equation

$$\dot{H} = -\frac{\varrho g r^4}{8\eta\,(a+b)\,R^2} H\,,$$

with a solution

$$H(t) = H_0 \exp\left(-\frac{\varrho g r^4}{8\eta\,(a+b)\,R^2} t\right)\,.$$

The pressure difference (with respect to atmospheric pressure) decreases linearly in the tube from $h\varrho g$ to zero, so at the manometer,

$$\Delta p_b = \frac{b}{a+b}\Delta p\,.$$

This pressure is further increased by capillary pressure

$$p_\mathrm{k} = \frac{2\sigma}{r}\,.$$

For the level of tea in the tube as a function of time, we get

$$h(t) = \frac{1}{\varrho g}\left(p_\mathrm{k} + \Delta p_b\right) = \frac{2\sigma}{r\varrho g} + \frac{b}{a+b} H_0 \exp\left(-\frac{\varrho g r^4}{8\eta\,(a+b)\,R^2} t\right)\,.$$

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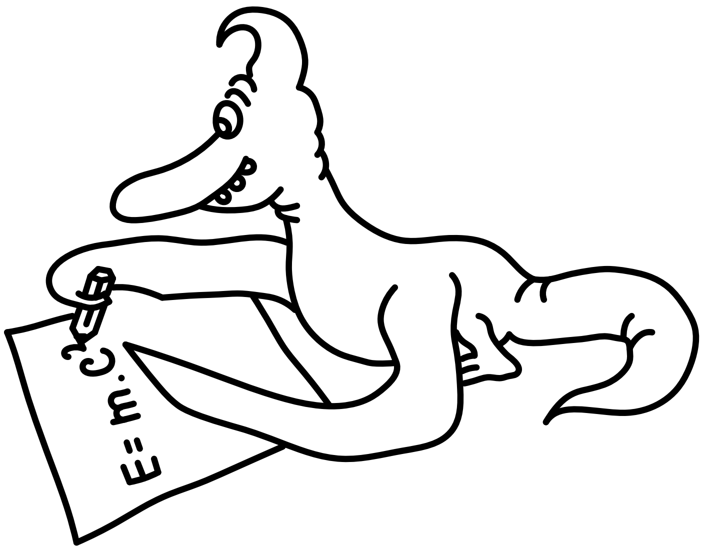

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