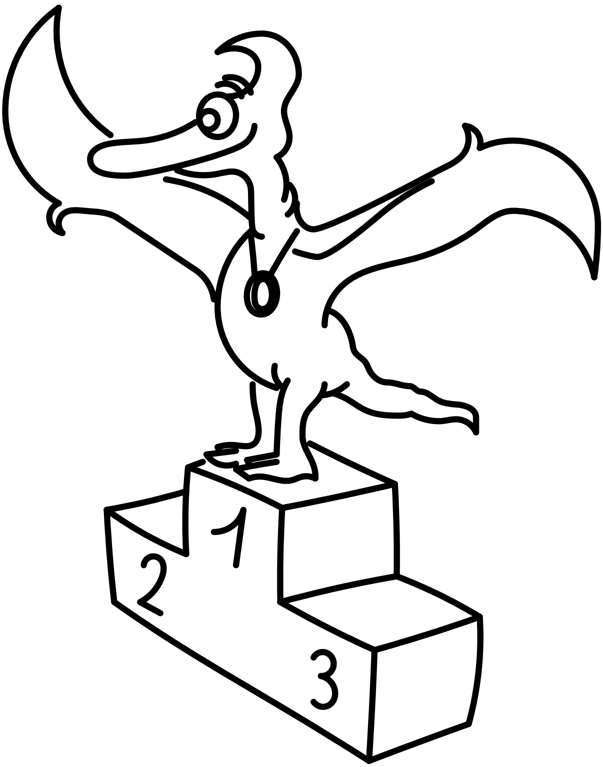
12^{th} FYKOS Physics Brawl
Solutions of problems



Problem AA ... wouldn't steal a car

We're downloading (legally) a film of size 2.1 GB*; the download speed is* $350\,\mathrm{kB{\cdot}s^{-1}}$*. The length of the film is* 90 min*. How long after the download begins can we start watching the film to be able to watch it without pausing?* *Mirek didn't want to pay up (for faster download speed).*

The download time is $2.1\,\mathrm{GB}/350\,\mathrm{kB{\cdot}s^{-1}} = 100\,\mathrm{min}$. We can start watching the film 90 min before the end of the download. Therefore we have to wait at least $100\,\mathrm{min} - 90\,\mathrm{min} = 10\,\mathrm{min}$ after the download begins to watch the film continuously.

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Problem AB ... simple conveyor belt

We have a conveyor belt of mass M (by itself, without any load). Initially (at time 0 s*), there is some sand with mass m lying on the conveyor belt; the sand gradually falls off the belt with a constant rate μ (mass of sand falling per unit time, in kilograms per second), from time* 0 s *until there is no sand left on the belt. What should be the force exerted on the conveyor belt at any time t (between* 0 s *and the moment when all sand falls off) so that it'd move with* ***constant acceleration*** *a?* *Karel simplified one complicated problem.*

First, let us remember Newton's second law of motion, which tells us that $F = m_{\mathrm{tot}}a$, where F is the exerted force and m_{tot} is the total mass of the body. In our case, the total mass is time dependent, but we can express it quite simply as $m_{\mathrm{tot}} = M + m - \mu t$, the reasoning being that $M + m$ is the initial mass and μt is the mass of sand that has fallen off the belt from the start up to time t. The force we're looking for is therefore $F = (M + m - \mu t)\, a$.

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Problem AC ... dropped banknote

Would it be worth it for Bill Gates to pick up a \$100 banknote from the ground? Let's say that his annual income is 30,000,000,000 *Czech crowns and that by picking up the note, his average income would be stopped for 3 seconds. How much money (in Czech crowns) could he earn by picking up the banknote? In case he'd be losing money that way, give the result as a negative number. Use the exchange rate* 1 CZK = \$0.046*. His income is continuous and constant.* *Matěj stole this from someone, no idea who.*

After conversion to US Dollars, the income of Bill Gates is \$1,380,000,000. This value divided by the number of seconds in a year equals the average income $\$43.7\,\mathrm{s^{-1}}$. That makes \$131.3 lost every 3 seconds when Bill is not working. Now we can see that he shouldn't pick up a \$100 banknote, because it would drop his earnings by \$31.3, or 680 CZK.

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Problem AD ... zero to hundred

One of the performance characteristics of a car is the time needed to accelerate from $0\,\mathrm{km\cdot h^{-1}}$ *to* $100\,\mathrm{km\cdot h^{-1}}$. *Consider a car that can do it in* $4.0\,\mathrm{s}$. *Find the distance travelled by the car while accelerating. Suppose that the acceleration of the car is constant.*

Karel was wondering about accelerating.

The distance s covered by a uniformly accelerating object during time t can be expressed as $s = at^2/2$, where a denotes the (constant) acceleration. In our case, the acceleration is

$$a = \Delta v/\Delta t = 100/4/3.6\,\mathrm{m\cdot s^{-2}} = 6.94\,\mathrm{m\cdot s^{-2}}\,.$$

We can now find the distance

$$s = \frac{1}{2}\Delta v \cdot \Delta t \doteq 55.6\,\mathrm{m}\,.$$

Assuming the acceleration is constant, the car travels a distance of $56\,\mathrm{m}$.

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Problem AE ... bathing

When Dano fills his bathtub, he first turns on the hot water with temperature $t_1 = 52\,\mathrm{^\circ C}$. *Because the hot water is too hot, he then mixes in some cold water with temperature* $t_2 = 18\,\mathrm{^\circ C}$. *He wants to achieve a temperature* $t = 42\,\mathrm{^\circ C}$ *(which is not ideal, but some people like it hot). What is the ratio of water volumes* $K = V_2/V_1$ *needed to achieve this temperature? Neglect the heat capacity of the bathtub and any thermal exchange with the surroundings. Volume* V_1 *labels the water with temperature* t_1.

Karel likes to have a bath.

We could write down the calorimetry formula, but can manage to solve this problem just with a simple argument. Let us assume that the specific heat of water is constant in the temperature range from t_2 to t_1 and that the volume of the water also doesn't depend on the temperature. Then, the final temperature after mixing can be written as the average of cold and hot water temperatures weighted by their respective volumes,

$$t = \frac{V_1 t_1 + V_2 t_2}{V_1 + V_2}\,.$$

Through simple manipulations, we rewrite the equation using the ratio V_2/V_1,

$$t\,(V_1 + V_2) = V_1 t_1 + V_2 t_2 \qquad \Rightarrow \qquad K = \frac{V_2}{V_1} = \frac{t - t_1}{t_2 - t} = \frac{5}{12} \doteq 0.42\,.$$

The volume ratio is $K = 0.42$.

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Problem AF ... comparison of the calorific value

What is the difference between heat of combustion of one litre of diesel fuel and one litre of petrol (gasoline)? The heat of combustion of diesel is $H_\mathrm{d} = 42.6\,\mathrm{MJ\cdot kg^{-1}}$ and that of petrol is $H_\mathrm{p} = 43.6\,\mathrm{MJ\cdot kg^{-1}}$. The density of petrol is $\varrho_\mathrm{p} = 740\,\mathrm{kg\cdot m^{-3}}$, the density of diesel is $\varrho_\mathrm{d} = 840\,\mathrm{kg\cdot m^{-3}}$. The answer must contain two information: ***which fuel*** *has the higher heat of combustion per litre and by* ***how much****.*

Karel was thinking about the efficiency of diesel fuel.

Since we know the heats of combustion per unit mass, all we have to do is multiply these values by densities of respective fuels to get the heats of combustion per unit volume. The only catch in this problem is that the densities are given in kilograms per cubic meter, so we have to use the conversion $1\,\mathrm{l} = 10^{-3}\,\mathrm{m^3}$. After multiplication, we obtain

$$E_\mathrm{d} = H_\mathrm{d}\varrho_\mathrm{d} \doteq 35.8\,\mathrm{MJ\cdot l^{-1}}\,, \quad E_\mathrm{p} = H_\mathrm{p}\varrho_\mathrm{p} \doteq 32.3\,\mathrm{MJ\cdot l^{-1}}\,,$$

$$E_\mathrm{d} - E_\mathrm{p} = (H_\mathrm{d}\varrho_\mathrm{d} - H_\mathrm{p}\varrho_\mathrm{p}) \doteq 3.5\,\mathrm{MJ\cdot l^{-1}}$$

Diesel has higher heat of combustion per litre than gasoline, namely by $3.5\,\mathrm{MJ\cdot l^{-1}}$.

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Problem AG ... conveyor belt travelling problem

At the Dubai airport, there are conveyor belts for walking; the velocity of a conveyor belt is v. Matěj needs to get to the terminal B27. He can either go directly to the terminal without using conveyor belts or take an alternative path where he gets to walk on conveyor belts. The second path is 1/3 longer, but half of its length is covered by conveyor belts. After a few minutes of hard calculations, he finds out that both paths take the same time, so he decides to take the moving walkway because it is more fun. Your task is to find Matěj's walking speed u (assume that this speed is constant).

Matěj loves walking on conveyor belts.

First, let us denote the length of the direct path by s. Then, this path takes time

$$t = \frac{s}{u}\,.$$

The other path consists of two parts. The first part has length $\frac{2}{3}s$ and is travelled at velocity u; the second part has length $\frac{2}{3}s$ and is travelled at velocity $v+u$. The total time can be expressed as

$$t = \frac{2}{3}\frac{s}{u} + \frac{2}{3}\frac{s}{v+u}\,.$$

Because the two times are equal, we can solve the equation for u,

$$\frac{s}{u} = \frac{2s}{3u} + \frac{2s}{3(v+u)}\,,$$

$$\frac{1}{3u} = \frac{2}{3(v+u)}\,,$$

$$u = v\,.$$

Matěj's walking speed is the same as the speed of a conveyor belt.

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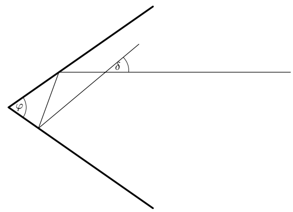
Problem AH ... crossed mirrors

A ray of light is being reflected by two planar mirrors (see figure). The angle between the mirrors is φ. How does the angle δ (see figure) depend on the angle φ? Daniel was studying optics.



Denoting the angles between the ray and the first and second mirror α and β respectively, we have the following relation for the left triangle $\alpha + \beta + \varphi = \pi$. For the right triangle, we get

$$2\left(\frac{\pi}{2} - \alpha\right) + 2\left(\frac{\pi}{2} - \beta\right) + \delta = \pi\,,$$

from which we can easily express $\delta = 2\,(\alpha + \beta) - \pi$. The last step is to substitute for $\alpha + \beta$ from the first expression, which leads to $\delta = \pi - 2\varphi$.

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Problem BA ... antimatterianism

Perhaps you've heard stories of so-called breatharians and pranists, who only need to breath to "suck out" the life energy from their surroundings. However, what if someone wanted to survive only on antimatter? That is, it's apparently impossible, but we're looking for an estimate of the minimum mass of antimatter such a human would have to carry inside to survive until natural death. Consider a 20-year-old adult who's expected to die in 60 years (i.e. at the age of 80). The power he needs to survive is $P = 100\,\mathrm{W}$. He has plenty of matter available; also let's assume that he consumes the energy with $100\,\%$ efficiency.

Karel was wondering how much a breatharian steals from the fridge at night.

The well-known (and often misunderstood) equation $E = mc^2$ says that matter (or antimatter) with mass m has an equivalent energy of mc^2. Reactions of antimatter with matter will lead to annihilation and release the energy of the antimatter and also that of the matter. So when we have antimatter with mass m and plenty of matter, we can gain energy $2mc^2$ (the reaction ratio is 1 to 1).

Power P over time $t = 60\,\mathrm{yr} \doteq 1.9 \cdot 10^9\,\mathrm{s}$ equals energy $E = Pt$, which in combination with $E = 2mc^2$ leads to

$$m = \frac{Pt}{2c^2} \doteq 1 \cdot 10^{-6}\,\mathrm{kg} = 1\,\mathrm{mg}\,.$$

This is a minuscule amount in comparison to the weight of an average human body. However, to obtain 1 mg of antimatter in current particle accelerators, we would have to let them run for about 100 million years. Which is a lot.

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Problem BB ... interplanetary pendulum

Let's denote the ratio of Earth's mass to Moon's mass by a. Next, let's denote the respective ratio of radii by b. $a > 1$& $b > 1$. Find the ratio of periods of the same pendulum on Earth's surface and on Moon's surface (in terms of a and b only).

Matěj loves the song Organism Do Evolve.[1]

The period of a simple pendulum can be expressed as

$$T = 2\pi\sqrt{\frac{l}{g}}\,.$$

Was this a physical, not mathematical, pendulum, the right-hand side would just be multiplied by a constant dependent on the moment of inertia. The only thing we need to know here is the ratio of gravitational accelerations on the surface of the Earth and the Moon. Gravitational acceleration is given by the formula

$$g = G\frac{M}{R^2}\,.$$

So, the ratio of those two gravitational accelerations is $\frac{g_{\mathrm{E}}}{g_{\mathrm{M}}} = \frac{GM_{\mathrm{E}}R_{\mathrm{M}}^2}{GM_M R_{\mathrm{E}}^2} = \frac{a}{b^2}$, where the subscripts stand for the Earth and the Moon.

Then, the ratio of the periods of the given pendulum is

$$\frac{T_{\mathrm{E}}}{T_{\mathrm{M}}} = \sqrt{\frac{b^2}{a}} = \frac{b}{\sqrt{a}}\,.$$

For Earth and Moon, this amounts to approximately 0.41.

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Problem BC ... charged balls

There are two rigid (inelastic) strings of negligible weight with lengths $l = 1.0\,\mathrm{m}$ attached to the same point at height $h = 3.0\,\mathrm{m}$ above the ground. First, we connect one small metal ball with mass m at the free end of one string. The charge of the ball is $Q = 2.8\,\mathrm{\mu C}$. Then, we attach an identical ball with the same charge at the free end of the other string. The balls electrically repel each other; they reach an equilibrium position when their height above the ground increases by $\Delta h = 3.0\,\mathrm{cm}$. What is the mass of one ball? *Danka is fascinated by charged balls.*

After reaching the equilibrium position, the angle between the strings is α. The tensile force of each string responsible for keeping each ball stable compensates the effect of gravitational and electrostatic forces. The gravitational force acting on each ball is

$$F_{\mathrm{g}} = mg\,.$$

The repulsive electrostatic force depends on the charge of the balls and also on their mutual distance d,

$$F_e = \frac{1}{4\pi\varepsilon}\frac{Q^2}{d^2}\,.$$

[1] https://www.youtube.com/watch?v=dx-Sy5M3bME

The distance d follows from the geometry of the problem

$$\left(\frac{d}{2}\right)^2 = l^2 - (l - \Delta h)^2 \qquad \Rightarrow \qquad d = 2\sqrt{2l\Delta h - (\Delta h)^2}\,.$$

Since the tensile force compensates the force resulting from F_g and F_e, it must have the opposite direction, so the angle α can be expressed as

$$\tan\alpha = \frac{F_e}{F_g}\,.$$

Other expressions for this angle follow from simple geometry:

$$\cos\alpha = \frac{l - \Delta h}{l}\,, \quad \sin\alpha = \frac{d}{2l}\,.$$

So,

$$\tan\alpha = \frac{d}{2(l - \Delta h)}\,.$$

Using the formulas above, we can write

$$\frac{d}{2(l - \Delta h)} = \frac{F_e}{F_g}\,,$$
$$\frac{d}{2(l - \Delta h)} = \frac{Q^2}{4\pi\varepsilon m g d^2}\,.$$

Finally, the mass m is

$$m = \frac{Q^2(l - \Delta h)}{2\pi\varepsilon g d^3}\,,$$
$$m = \frac{Q^2(l - \Delta h)}{16\pi\varepsilon g(2l\Delta h - (\Delta h)^2)^{\frac{3}{2}}} \doteq 120\,\mathrm{g}\,.$$

The mass of each small metal ball is approximately 120 g.

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Problem BD ... the new dimension

Use dimensional analysis to determine the formula for dynamic viscosity η of a material using only the Boltzmann constant k_B, absolute temperature T, collision radius r and molecular mass m; let's denote the dimensionless constant of the formula by C.
Hint: *The SI unit of dynamic viscosity is* Pa·s.

Tomáš beheld the beauty of nature without constants.

We are looking for a dynamic viscosity formula in the form $\eta = C k_\mathrm{B}^\alpha T^\beta m^\gamma r^\delta$. Writing down the SI base units of individual physical quantities, we get the equation

$$\frac{\mathrm{kg}}{\mathrm{m{\cdot}s}} = C\left(\frac{\mathrm{kg{\cdot}m^2}}{\mathrm{s^2{\cdot}K}}\right)^\alpha \mathrm{K}^\beta \mathrm{kg}^\gamma \mathrm{m}^\delta\,.$$

By comparing powers of respective units on the left-hand and right-hand side, we obtain a system of four linear equations

$$\begin{aligned} 1 &= \alpha + \gamma\,, \\ 0 &= -\alpha + \beta\,, \\ -1 &= 2\alpha + \delta\,, \\ -1 &= -2\alpha\,. \end{aligned}$$

This system can be easily solved to obtain the desired formula

$$\eta = C\frac{\sqrt{k_\mathrm{B} T m}}{r^2}\,.$$

As you can see, we have obtained a valid formula for dynamic viscosity without any complex derivations. This method will never determine the value of the constant C, but that can be measured experimentally.

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Problem BE ... repulsive cube

Consider a solid homogeneous metal cube with side of length a. The electrical resistance between two opposite faces of the cube is R. What will be the resistance of a cube from the same material in the same configuration, but with sides of length b? The cube is connected to the circuit using two perfectly conducting plates in perfect contact with the two opposite faces of the cube.

Karel was casting dice.

We are going to solve this problem with just a little bit of insight. On one hand, the resistance depends linearly on the length of the conducting body, therefore it would be multiplied by b/a if we only extended the cube in one direction; on the other hand, it is inversely proportional to the cross section of the body, so extending the body in the two remaining dimensions will multiply the resistance by $(a/b)^2$.

Thus, the resistance of a cube with side length b is given by

$$R' = \frac{a}{b}R\,,$$

where R is the resistance of a cube with side length a. Expansion of a cube leads to a decrease of its resistance.

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Problem BF ... bubble bath for Janap

Janap wanted to dive into a bubble bath with temperature at least $T_2 = 35\,^\circ\mathrm{C}$. However, the maximum temperature of tap water was only $T_1 = 18\,^\circ\mathrm{C}$. Janap thought that it would be interesting to heat some water by dropping it from an airplane at certain altitude h and converting its kinetic energy to heat. Assume that the water is at rest when dropped from the airplane and that all the kinetic energy acquired during the fall is used to heat up the water. What should *the altitude of the airplane be to heat the water just right? Assume that the gravitational acceleration is $g = 9.81\,\mathrm{kg{\cdot}m^{-2}}$. You can also make use of some of these constants: specific heat of water $c = 4{,}200\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$, and gravitational constant $G = 6.67\cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$, Earth's radius $R_\mathrm{Z} = 6378\,\mathrm{km}$, Earth's mass $M_\mathrm{Z} = 5.97\cdot 10^{24}\,\mathrm{kg}$. Volume and mass of the bath and Janap will not be disclosed.*

Karel remembered Janap.

The potential energy of a mass m at altitude h is

$$E \approx mgh\,.$$

The heat required to warm up the water to the desired temperature is

$$Q = mc\Delta T\,,$$

where $\Delta T = T_2 - T_1$. This heat should be equal to the difference in potential energy,

$$mgh = mc\Delta T\,.$$

From this equation, we obtain the altitude of the plane

$$h = \frac{c\Delta T}{g} \doteq 7{,}300\,\mathrm{m}\,.$$

At altitude h, the gravitational acceleration is $g' = \frac{GM_\mathrm{Z}}{(R_\mathrm{Z}+h)^2} = 9.77\,\mathrm{kg{\cdot}m^{-2}} \approx g$, meaning our simplified model is accurate enough. The water has to be released at altitude $h = 7{,}300\,\mathrm{m}$.

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Problem BG ... mercury column

A thin cylindrical tube with length $l = 1.00\,\mathrm{m}$ is half-way submerged (vertically) in a basin with mercury of density $\varrho = 13.6\cdot 10^3\,\mathrm{kg{\cdot}m^{-3}}$. We close the top end of the tube and lift it upwards. Part of the mercury spills out of the tube. What will be the height of the column of mercury remaining in the tube? The atmospheric pressure is $p_\mathrm{a} = 101\,\mathrm{kPa}$.
Hint: *Consider an isothermal expansion.*

Vašek couldn't make up an origin for this problem.

The mercury column in the tube is affected by the gravitational force F_g, force F_a induced by atmospheric pressure and by the force F_i induced by air pressure in the tube. At static equilibrium, these forces satisfy the equation

$$F_\mathrm{g} + F_\mathrm{i} = F_\mathrm{a}\,.$$

Force induced by pressure is given by the product of pressure and area, thus we have

$$mg + p_\mathrm{i}S = p_\mathrm{a}S\,,$$

where m is the mass of mercury in the tube, g is the gravitational acceleration, p_i is the pressure of the gas in the tube and S is the inner cross-sectional area of the tube. The mass can be expressed as $m = Sx\varrho$, where x is the equilibrium height of the mercury column. Simple modifications of the force balance equation result in

$$p_\mathrm{a} - p_\mathrm{i} = \varrho x g\,. \tag{1}$$

When the tube is lifted, the gas trapped inside expands isothermally. By Boyle's law, it holds that

$$p_\mathrm{a} S \frac{l}{2} = p_\mathrm{i} S\left(l - x\right)\,.$$

Hence the pressure is

$$p_\mathrm{i} = p_\mathrm{a} \frac{l}{2\left(l - x\right)}\,.$$

This expression can be substituted into equation (1) to obtain a quadratic equation in x

$$2\varrho g x^2 - 2\left(p_\mathrm{a} + l\varrho g\right)x + p_\mathrm{a} l = 0\,.$$

The physically meaningful solution to this equation is

$$x = \frac{1}{2}\left(l + \frac{p_\mathrm{a}}{\varrho g} - \sqrt{\left(\frac{p_\mathrm{a}}{\varrho g}\right)^2 + l^2}\right)\,.$$

For the given values, the height of the mercury column is $x = 0.25\,\mathrm{m}$.

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Problem BH ... sunken

We place an empty cylindrical thin-walled container into a water channel. The height of the container is $h = 15\,\mathrm{cm}$, the radius of its base is $r = 5.0\,\mathrm{cm}$ and its mass is $m = 0.5\,\mathrm{kg}$. The stream in the channel carries the container with constant velocity $v = 2.0\,\mathrm{m{\cdot}s^{-1}}$. There is a hole at the bottom of the container, so $V_0 = 25\,\mathrm{ml}$ of water flows inside it each second. Calculate the length in metres travelled by the container before it sinks under the water surface.

Danka was contemplating the amount of waste in the rivers.

The volumetric flow into the container is $Q_v = V_0\,\mathrm{s}^{-1}$. The container sinks when the buoyant force reaches its maximum value. Up to that moment, buoyancy and weight cancel out, so

$$(m + \varrho Q_v t)g = \pi r^2 h \varrho g$$

where ϱ is the density of water and t is time (measured from the point when the container started leaking water). We can express

$$t = \frac{\pi r^2 h \varrho - m}{Q_v \varrho}\,.$$

Then, the distance travelled by the container is

$$s = vt \doteq 54.25\,\mathrm{m}\,.$$

The container will be carried 54 m away from its starting point before it sinks.

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Problem CA ... walk on a boat à la Dan

Daniel, just like a proper rich man, owns a yacht. The yacht isn't the most luxurious and largest, but it's still pretty large. Its displacement is $M = 40\,\mathrm{tons}$ (of water) and it's $D = 20\,\mathrm{m}$ long. Let's imagine that the yacht is in a tank with calm water and at rest with respect to the edge of the water tank; Daniel is standing at rest at its front. By how much does the yacht move with respect to the water tank if Daniel slowly walks from its front to back? Daniel didn't want to tell us his exact weight but assume it's $m = 70\,\mathrm{kg}$. Compute the result to at least two decimal places. *Karel was looking at problems on Brilliant.*

The water displacement of the yacht corresponds to its weight. So the mass of the yacht is M. Because the system was initially at rest, the overall momentum must also be zero during the movement and at the end of the movement. Dan walks from the nose to the stern so the ship will move in the opposite direction. If we assume that Daniel is moving with a velocity v, then the conservation law of momentum ($0 = mv + MV$) means that the ship will move at velocity $V = -mv/M$. I. e. the direction is opposite and the ship's velocity is inversely proportional to the mass of the ship.

We just have to realize that the resulting shift is independent of Daniel's velocity. Daniel walks the same distance in any case. If we think about it, the position of the centre of mass of the ship-Daniel system stays constant and Daniel will move with respect to the tank by $d_1 = MD/(M + m)$. Therefore, the ship moves $d_2 = mD/(M + m) \approx mD/M = 0.035\,\mathrm{m} = 3.5\,\mathrm{cm}$ w.r.t. the tank.

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Problem CB ... perpetuum mobile of the second kind

Let's assume that the 2nd law of thermodynamics doesn't hold (so heat can flow from a colder body to a warmer one). Consider a boat that, to run, draws water with temperature $T_1 = 3.2\,^\circ\mathrm{C}$ from the sea, cools it down and drops out an ice cube with temperature $T_2 = -5.0\,^\circ\mathrm{C}$ each minute. How long should an edge of the cube be if the power consumption of the ship's motor is $P = 0.8\,\mathrm{MW}$? *Viktor was bored at a thermodynamics lecture.*

Let's denote the edge of the cube as a, specific heat capacity of water as $c_\mathrm{v} = 4{,}200\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$, specific heat capacity of ice as $\varrho_\mathrm{l} = 917\,\mathrm{kg{\cdot}m^{-3}}$ and specific heat of melting as $l = 334\,\mathrm{kJ{\cdot}kg^{-1}}$. During each time interval $t = 1\,\mathrm{min} = 60\,\mathrm{s}$, the ship gains energy

$$E = mc_\mathrm{v}(T_1 - T_\mathrm{t}) + ml + mc_\mathrm{l}(T_\mathrm{t} - T_2)$$

from the water of mass $m = \varrho_\mathrm{l} a^3$. This is done by cooling the water to $T_\mathrm{t} = 0^\circ\mathrm{C}$, freezing ice and cooling ice to $T_2 = -5.0\,^\circ\mathrm{C}$.

The energy acquired over time must be equal to the power consumption required, i.e.

$$P = \frac{E}{t} = \frac{\varrho_\mathrm{l} a^3}{t}\left(c_\mathrm{v}(T_1 - T_\mathrm{t}) + l + c_\mathrm{l}(T_\mathrm{t} - T_2)\right) ,$$

$$a = \sqrt[3]{\frac{Pt}{\varrho_\mathrm{l}}\left(c_\mathrm{v}(T_1 - T_\mathrm{t}) + l + c_\mathrm{l}(T_\mathrm{t} - T_2)\right)^{-1}} .$$

By assigning the given values we get $a \doteq 53\,\mathrm{cm}$.

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Problem CC ... asteroids

There is an asteroid orbiting a star. Far away, there is another asteroid orbiting another star. The orbital radius and orbital period of the second asteroid are three times larger. Determine the ratio of stellar masses (the mass of the first star to the mass of the second star).

Matěj likes ratios.

We start from balance of centrifugal and gravitational forces

$$\frac{GM}{R^2} = m\omega^2 R\,,$$

where M is the mass of the star, m is the mass of the planet, R is the orbital radius, G is the gravitational constant and $\omega = \frac{2\pi}{T}$ is the angular velocity of the orbiting planet. We are able to express the mass of the star

$$M = \frac{4\pi^2 R^3}{GT^2}\,.$$

Although we have derived this relationship "only" for circular orbits, Kepler's third law tells us that it holds even for elliptical orbits. We now know that the mass of each star is directly proportional to the cube of orbital radius and inversely proportional to the square of orbital period. When the radius and period both triple, the mass must triple too, so the ratio of masses is 1:3.

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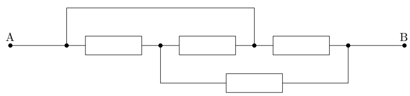
Problem CD ... a complex network

What will be the total resistance between nodes A and B in the circuit depicted in the figure if each resistor has the same resistance R?



Karel was trying out how good he is at solving physics problems.

It might seem that the circuit contains some loop that would require the use of Kirhoff's laws, or alternatively use some triangle-star transformation, but on the contrary; if the network is correctly redrawn, we find that it is a relatively simple network of resistors connected in parallel and in series, whose resistance we can calculate easily. The original network is the same as the one in figure 1. We continue to modify the network according to basic rules.

As we can see from figure 2, the result is $3R/5 = 0.6\,R$.

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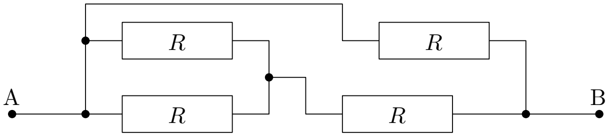

Fig. 1: Modified circuit

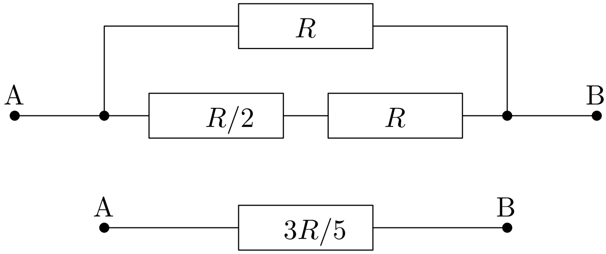

Fig. 2: Modified circuit

Problem CE ... small data

Matěj decided to test a die. He started throwing it and writing down the numbers that came up. After nine throws, he started analysing the gained data. He found out that the minimum number was 1 and the maximum was 6. The median of the set was 4, the (unique) mode was 2 and the arithmetic mean was 3.4 *when rounded to one decimal place. What was the geometric mean?* *Matěj likes gambling.*

After closer inspection, we find that all 9 values Matěj measured are uniquely determined by the information provided.

We know that each of numbers 1, 4 and 6 occurs at least once. Next, there are at least three 2-s (because 2 is the **unique** mode). Because the median is 4, there cannot be more than four numbers smaller than 4. This means that there are just three 2-s. Now, we don't know three numbers; each of them could be 4, 5 or 6. The value of the arithmetic mean tells us that the sum of all numbers is exactly 31 (30 or 32 would be rounded to 3.3 or 3.6 respectively). The missing numbers could be 4, 4, 6 or 4, 5, 5. We can't have three 4-s because the mode is unique.

The measured values are therefore:

$$1, 2, 2, 2, 4, 4, 5, 5, 6\,.$$

Now, we just calculate the value of the geometric mean $\sqrt[9]{1 \cdot 2 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \cdot 6} = 2.99$.

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Problem CF ... hard-working heart

How much work does the human heart do in a lifetime? The pressure amplitude (the difference between systolic and diastolic pressure) is about $\Delta p = 40\,\mathrm{mmHg}$ *(mm of mercury column). Assume that a human lives on average* $T = 82\,\mathrm{years}$ *and the average volumetric flow rate* *through the heart is roughly* $Q = 6\,\mathrm{l{\cdot}min^{-1}}$.

Karel was inspired by Brilliant, where they seemed to get it wrong.

In SI base units, we have $\Delta p = 5.33 \cdot 10^3\,\mathrm{Pa}$, $T = 2.59 \cdot 10^9\,\mathrm{s}$ and $Q = 1 \cdot 10^{-4}\,\mathrm{m^3{\cdot}s^{-1}}$. Let's assume the heart needs to exert a force F over a distance s. The work is given by the formula $W = Fs = S\ \Delta ps\ = V\ \Delta p = Qt\Delta p$, where we consider S to be the effective cross-sectional area of chambers of the heart or the aorta; the total volume of blood it needs to push through is $V = Ss = Qt$. After computing the result for the given values, we get $W \approx 1.4 \cdot 10^9\,\mathrm{J}$.

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Problem CG ... I'm just all right

What would be the temperature of the Earth if we considered it a perfect black body which gains energy from solar radiation and doesn't contain any heat source of its own? The power radiated by the Sun is $3.827 \cdot 10^{26}\,\mathrm{W}$, *and its distance from the Earth is* $1.5 \cdot 10^{11}\,\mathrm{m}$. *Assume that Earth is a perfect sphere with constant temperature on its entire surface. Calculate the result in degrees Celsius.* *Štěpán loves classic problems.*

The radius of the Earth is r and its distance from the Sun is R. The Earth is always illuminated from one side. The Sun shines just on the (cross-sectional) area πr^2.

We can say that the power of incident solar radiation is $P_S = P\frac{r^2}{4R^2}$, because $r \ll R$.

The power of the Earth's radiation is $P_\mathrm{Z} = \sigma T^4 {\cdot} 4\pi r^2$, where σ denotes the Stefan-Boltzmann constant.

Both of these powers have to be the same because the temperature is constant. From this, we are able to express the thermodynamic temperature T. We can see the radius of the Earth is irrelevant.

$$T = \sqrt[4]{\frac{P}{16\pi R^2 \sigma}} = 277.9\,\mathrm{K} = 4.8\,^\circ\mathrm{C}\,.$$

The constant temperature on Earth would be $4.8\,^\circ\mathrm{C}$.

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Problem CH ... lighthouse

There's an isotropic light source with luminous intensity $I_1 = 100\,\mathrm{cd}$ *located in the focal point of a parabolic mirror. The edge of the mirror (a cut through a paraboloid, perpendicular to its axis of symmetry) is circular, contains the focal point in its centre and has radius* $R = 1\,\mathrm{m}$. *At a distance* $L = 1\,\mathrm{km}$ *from the mirror (measured along the axis of the paraboloid), there's a circular target with radius* R. *How many times does the effective brightness of the lighthouse (as measured by the target) increase when the mirror is added?*

Kuba wanted to estimate the importance of the mirror in a lighthouse.

The ratio of fluxes incident on the target surface is equal to the ratio of solid angles formed by rays that reach the target. Without the mirror, we can write (approximately for $R \ll L$)

$$\Omega_1 = \frac{\pi R^2}{L^2} = \frac{\pi R^2}{L^2}\,.$$

The mirror contributes with an additional half-space, i. e. solid angle $\Omega_2 = 2\pi$. For the ratio of incident fluxes, we have

$$\frac{I_2}{I_1} = \frac{\Omega_1 + \Omega_2}{\Omega_1} \approx \frac{\Omega_2}{\Omega_1} = \frac{2L^2}{R^2} \doteq 2.00 \cdot 10^6 \,.$$

The result is a two-million-fold increase in brightness.

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Problem DA ... Bernoulli in real life

Water flows from a tap with a nozzle of radius $R = 0.005\,\mathrm{m}$ at a (volumetric) flow rate $Q_0 = 2\cdot 10^{-5}\,\mathrm{m^3\cdot s^{-1}}$. At what distance below the nozzle does the continuous stream start breaking into droplets? Consider water to be an ideal liquid. Droplets start forming when the stream has radius smaller than $r = 0.003\,\mathrm{m}$. *Matěj almost drowned.*

Thanks to gravity, the stream will accelerate on its way down. The velocity of the stream at the top is

$$v_0 = \frac{Q_0}{\pi R^2} \,.$$

At depth h below the tap, the law of conservation of energy gives

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = hg \,,$$

$$v = \sqrt{2gh + v_0^2} \,.$$

We could obtain the same result from the Bernoulli equation, assuming the pressure in the stream is equal to the (constant) atmospheric pressure.

From the continuity equation, we get the depth corresponding to the critical radius

$$\pi r^2 v = Q_0 \,,$$

$$\pi r^2 \sqrt{2gh + \frac{Q_0^2}{\pi^2 R^4}} = Q_0 \,,$$

$$h = \frac{Q_0^2}{2\pi^2 g r^4} - \frac{Q_0^2}{2\pi^2 g R^4} = \frac{Q_0^2}{2\pi^2 g}\left(\frac{1}{r^4} - \frac{1}{R^4}\right) \doteq 0.0222\,\mathrm{m} \,.$$

The stream begins to split apart at about 2 cm below the tap.

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Problem DB ... thunderstorm

You're casually running on train tracks in the evening with velocity $v = 15\,\mathrm{km\cdot h^{-1}}$ when you suddenly notice the dark shape of a train on the horizon. The train is moving right towards you with velocity $u = 160\,\mathrm{km\cdot h^{-1}}$. The conductor has already noticed you and sounded the horn, which emits a sound signal with frequency $f = 1{,}000\,\mathrm{Hz}$. What will be the frequency f' you *hear, if you're also running against the wind (the wind velocity is $w = 100\,\mathrm{km\cdot h^{-1}}$)? The speed of sound is $c = 340\,\mathrm{m\cdot s^{-1}}$.* *Tomáš didn't want to jump under a train.*

As a result of the mutual movement of the source and the observer of the wave, the individual waves are densified and the frequency changes. This effect is known as the Doppler phenomenon, and it tells us how frequency depends on the velocity of the observer and the source

$$f' = f\frac{c + v_\mathrm{p}}{c - v_\mathrm{z}}\,,$$

where the original transmitted frequency is f, the velocity of the source is v_z, the velocity of the observer is v_p and the speed of propagation of the waves in the given environment is c. If both observer and source are moving towards each other, their velocities are positive (negative otherwise). In this case, however, we still have to count the wind that blows from the source to the observer at speed w and thus increases the speed of propagation of the waves to $c' = c + w$. The resulting frequency we hear is

$$f' = f\frac{c' + v_\mathrm{p}}{c' - v_\mathrm{z}} = f\frac{c + w + v}{c + w - u}\,.$$

After substituting in the given values, we get the result $f' \doteq 1{,}150\,\mathrm{Hz}$.

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Problem DC ... football problem

A football player (European football, soccer in the USA) is standing $d = 15.0\,\mathrm{m}$ in front of a goal with height $h = 2.50\,\mathrm{m}$. He kicks the ball directly towards the target at an angle $\alpha = 30.0^\circ$ with respect to the horizontal plane. How fast should he kick the ball so that it would hit the goal directly without bouncing? (The answer is a range of velocities.) Neglect air resistance and the size of the ball. *Danka is unable to hit the goal.*

We take the motion of the ball as a oblique throw. The centre of the coordinate system is at the point of the kick. We get the motion equations

$$x = vt\cos\alpha\,, \quad y = vt\sin\alpha - \frac{1}{2}gt^2\,.$$

In order to strike the goal, the y component must lie within the range of $0 \le y \le h$ when the x component is equal to d. We get the condition by eliminating time from the equations

$$0 \le d\tan\alpha - \frac{gd^2}{2v^2\cos\alpha^2} \le h\,.$$

The condition for initial velocity is

$$13.04\,\mathrm{m\cdot s^{-1}} \le v \le 15.46\,\mathrm{m\cdot s^{-1}}\,.$$

The football player has to kick the ball out at the speed larger than $13.0\,\mathrm{m\cdot s^{-1}}$ and smaller than $15.5\,\mathrm{m\cdot s^{-1}}$.

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Problem DD ... lens and mirror

In a room with height $h = 3\,\mathrm{m}$, we place a mirror on the floor underneath a light source. The light source is exactly at the ceiling level. At the height $5\,\mathrm{cm}$ above the mirror, we place horizontally a thin, positive (converging) lens with focal length $f = 30\,\mathrm{cm}$. How high above the ground will the image of the light source be? *Matěj played with optics.*

Let's denote the distance between the lens and the mirror $d = 5\,\mathrm{cm}$. Firstly, the lens displays the light source to a height h' above the ground (h' can be negative when the image is "under the ground"). Than, this image is reflected by the mirror so it's displayed to a height $-h'$. Finally, the image is displayed again by the lens (but in the other direction) to a height h''.

We can write the thin lens equation for the first image

$$\frac{1}{h-d} + \frac{1}{d-h'} = \frac{1}{f}\,,$$
$$h' = d - \frac{1}{\frac{1}{f} - \frac{1}{h-d}} \doteq -28.4\,\mathrm{cm}\,.$$

Than, the image is reflected by the mirror and we get second thin lens equation

$$\frac{1}{d-(-h')} + \frac{1}{h''-d} = \frac{1}{f}\,,$$
$$h'' = d + \frac{1}{\frac{1}{f} - \frac{1}{d+h'}} \doteq 18.14\,\mathrm{cm}\,.$$

The real image is 18 cm above the ground.

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Problem DE ... cleaning glasses

The tap of the kitchen sink has a turnable head, which is currently tilted at the angle $\alpha = 30^\circ$ with respect to the vertical. The head is $h = 30\,\mathrm{cm}$ above the bottom of the sink. The tap is set to form a single continuous stream with the diameter of the opening in the tap $d = 5\,\mathrm{mm}$, the volumetric flow then is $Q = 100\,\mathrm{ml{\cdot}s^{-1}}$. In this setting, the stream from the tap hits the bottom of the sink at horizontal distance l_1 (from the point right below the tap). If we set the tap to "shower" mode, where water exits through ten holes with diameters $d_2 = 1\,\mathrm{mm}$, the streams hit the bottom of the sink at horizontal distance l_2. Determine the ratio l_2/l_1. Assume that the distance between openings in the tap is negligibly small. *Mirek's glass was especially greasy.*

Assuming, with minimal loss of accuracy, that water is incompressible, the volumetric flow rate will remain constant as the area of the openings changes. Outflow velocity in the standard mode is

$$v_1 = \frac{4Q}{\pi d_1^2}\,,$$

after switching to the multiple streams mode, outflow velocity becomes

$$v_2 = \frac{4Q}{10\pi d_2^2}\,.$$

Knowing the initial velocity and height, we can determine the distance at which the stream hits the sink. The motion of the stream is determined

$$x = vt\cos\beta\,,\quad y = vt\sin\beta - \frac{1}{2}gt^2\,,$$

where we defined $\beta = \alpha - \pi/2$. Now setting $y = -h$ we can express x as a function of initial velocity as

$$x(v) = v^2\frac{\cos\beta}{g}\left(\sqrt{\frac{2gh}{v^2} + \sin^2\beta} + \sin\beta\right).$$

Calculating $l_1 = x(v_1) \doteq 16.178\,\mathrm{cm}$, $l_2 = x(v_2) \doteq 17.116\,\mathrm{cm}$ we get a ratio

$$\frac{l_2}{l_1} = 1.058\,.$$

The distance increases by a negligible 6 %. In the limit of decreasing opening sizes, the distance approaches $10\sqrt{3}\,\mathrm{cm}$.

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Problem DF ... four body problem

Four planets, each with the mass equal to the Moon's mass, orbit their common centre of gravity. The planets move with a stable circular orbit in such a way that they always form the vertices of a square with side length 384,400 km*. Find their orbital period. The Moon has mass* $7.35 \cdot 10^{22}$ kg*.* *Three planets aren't enough for Štěpán.*

For the planets to stay in orbit, the centrifugal force F_o and gravitational force F_G must be equal in magnitude.

Distance between the planets and the centre is $\frac{\sqrt{2}}{2}a$, where a is the length of the side of the square. Total centrifugal force at angular velocity $\omega = \frac{2\pi}{T}$, where T is the period, is $F_\mathrm{o} = m\omega^2 \cdot \frac{\sqrt{2}}{2}a$.

Gravitational force has to be calculated step by step. The force towards the neighbouring planets at distance a has magnitude $G\frac{m^2}{a^2}$, where G is the gravitational constant. These two forces are equal and perpendicular, so their sum is directed towards the centre and has magnitude $F_{G_1} = \sqrt{2}G\frac{m^2}{a^2}$. The planet opposite, at distance $a\sqrt{2}$, acts with force $F_{G_2} = G\frac{m^2}{2a^2}$ also directed towards the centre. Their sum is $F_G = F_{G_1} + F_{G_2} = G\frac{m^2}{a^2} \cdot \left(\sqrt{2} + \frac{1}{2}\right)$.

Expressing T from the equality $F_\mathrm{o} = F_G$

$$T = \sqrt{\frac{8\pi^2a^3}{Gm\left(4 + \sqrt{2}\right)}} \doteq 1.30 \cdot 10^7\,\mathrm{s} \doteq 150.4\,\mathrm{days}\,.$$

The period of one rotation in our system is 150.4 days.

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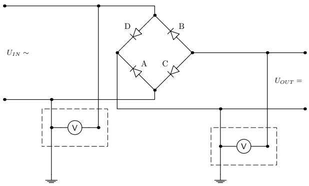
Problem DG ... Why is it smoking?

Lukáš built a full-wave rectifier and decided to test it (see figure). He took two oscilloscopes (since he knows the negative poles of one oscilloscope are connected…), connected one of them to the AC input and the other to the DC output. Unfortunately, the negative poles of both oscilloscopes were connected to the ground pin in the electrical socket. Which diode started smoking?



Lukáš burned down a germanium diode in the lab and was very surprised, since he used two oscilloscopes.

If we connect grounds of the oscilloscopes together, we see both poles of diode A will always be at the same potential, so we can replace the diode with a conductor. Diode D is connected directly between the terminals of the input voltage, so for one halfway of the input, it shorts the circuit and therefore carries very large current and burns out, so the magic smoke leaves diode D.

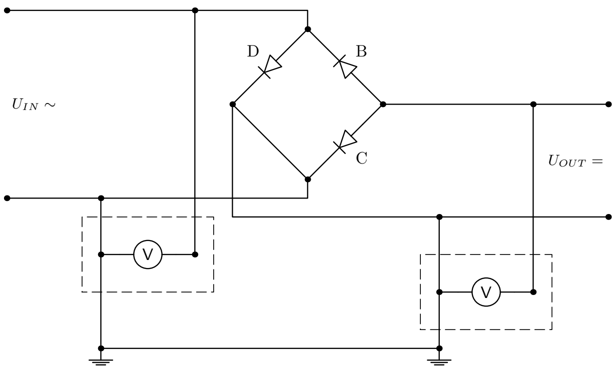


Fig. 3: Connected diodes.

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Problem DH ... solar sniper

Matěj suddenly finds himself on the equator at noon of the equinox, and he's sweating terribly. The sun is making him angry, so he pulls out his house-made weapon, which can shoot lightweight bullets at almost the speed of light, and fires a projectile directly towards the centre of the Sun. However, he forgot about the movement of celestial bodies. Find the minimum *velocity the bullet needs to have so that Matěj could hit the Sun.*

Matěj was thinking about the equator.

Matěj would hit the centre of the Sun if he wasn't in motion relative to it, but he did not consider that the Earth is in orbit around the Sun and rotates.

The velocity of rotation of the earth at the equator can be calculated easily, as we know the radius of the Earth $R_Z = 6380\,\mathrm{km}$ and the period of rotation $T_Z = 24\,\mathrm{h}$, being careful to convert the units, we get

$$v_Z = \frac{2\pi R_Z}{T_Z} = 464\,\mathrm{m\cdot s^{-1}}\,.$$

The orbital velocity v_S can be calculated similarly, we know the distance between the Earth and the Sun $R_S = 1\,\mathrm{AU} = 150 \cdot 10^9\,\mathrm{m}$, the orbital period is well known $T_S = 365.25\,\mathrm{days} = 3.16 \cdot 10^7\,\mathrm{s}$

$$v_S = \frac{2\pi R_S}{T_S} = 29{,}900\,\mathrm{m\cdot s^{-1}}\,.$$

We can readily see that the velocity v_Z is negligible, which makes our task much easier as we do not need to consider the direction in which the Earth rotates and the angle between the axis of rotation and the orbital plane.

Denoting the velocity of the projectile v, it will take it approximately $t = \frac{R_S}{v}$ to reach the Sun. The lateral distance the projectile travels in this time must be less than the radius of the Sun $r = 6.96 \cdot 10^8$ m or it will miss the Sun

$$\begin{aligned} v_S t &\leq r\,,\\ \frac{2\pi R_Z}{T_Z}\frac{R_S}{v} &\leq r\,,\\ v &\geq \frac{2\pi R_Z R_S}{T_Z r} = \frac{R_S}{r} v_S = 6.4 \cdot 10^6\,\mathrm{m\cdot s^{-1}}\,. \end{aligned}$$

We get the minimal velocity as approximately one fiftieth of the speed of light.

To get a more accurate result, we need to subtract the velocity due to rotation of the Earth from the orbital velocity, taking into account the tilt of axis of rotation relative to the axis of the orbital plane (approximately $\varphi = 23.4^\circ$). Subtracting the velocity vectors we get the velocity relative to the Sun

$$v' = \sqrt{(v_S - v_Z \cos\varphi)^2 + (v_Z \sin\varphi)^2}\,.$$

Plugging in the numbers we get

$$\begin{aligned} v' &= 29{,}440\,\mathrm{m\cdot s^{-1}}\,,\\ v &\geq \frac{R_S}{r} v' = 6.34 \cdot 10^6\,\mathrm{m\cdot s^{-1}}\,, \end{aligned}$$

which still is not completely accurate as the orbit of the Earth is elliptical and we do not know whether the problem asks about the spring or autumn equinox, so we cannot solve the problem exactly (but these corrections would be rather small).

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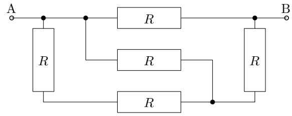
Problem EA ... those resistors again



Find the total electrical resistance between nodes A and B of the circuit shown in the figure. Each resistor has resistance R.

Karel was playing with IPE.

Let's mark the vertices in the circuit as C and D (see picture 4).

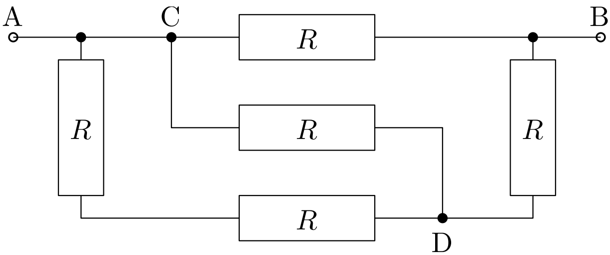


Fig. 4: Sketch of the circuit with the vertices labelled.

We can see that the configuration is quite unsuitable to be solved through adding together of parallel and serial resistors. We could use Kirchhoff's laws but we might get accidentally lost in the equations despite the circuit being relatively simple. One suitable and quite elegant simplification that we can use is the transformation from a triangle to a star. [2]. We modify the triangle BCD in this way. Instead of resistors with resistance R in the triangle, the star will have resistors with resistance $R/3$. You can see the modified circuit in figure 5

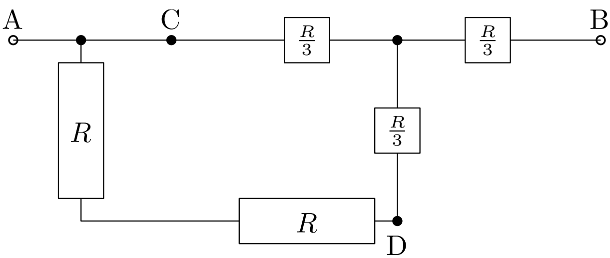


Fig. 5: Modified circuit

Now we can easily add resistors together and get the result

$$R_{\mathrm{tot}} = \frac{\frac{R}{3}\frac{7R}{3}}{\frac{R}{3}+\frac{7R}{3}} + \frac{R}{3} = \frac{5}{8}R = 0.625\,R\,.$$

[2] Derivation and examples of this transformation can be found in many textbooks covering electrical circuits, or on the internet.

The total resistance of the circuit is $0.625\,R$.

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Problem EB ... typical problem of a commuter

Matěj is walking to a bus stop along a straight road with velocity $v = 4\,\mathrm{km{\cdot}h^{-1}}$. He sees a bus arriving from the opposite direction to the bus stop with velocity $u = 36\,\mathrm{km{\cdot}h^{-1}}$, but doesn't know if it's bus number 201, which he needs to catch. At the moment when he can read number 201 on the bus, he immediately starts running with velocity $3v$. Will he make it? The resolution of the human eye is one arcminute and the display of the bus is readable if one can distinguish two pixels with distance $2\,\mathrm{cm}$. The deceleration of the bus is constant and it waits at the bus stop for $10\,\mathrm{s}$. Compute the time margin (how much time he can lose and still catch the bus; if he can't make it, this time should be negative). Initially, Matěj and the bus are both $50\,\mathrm{m}$ from the bus stop. *Matěj barely made it.*

Firstly, let's calculate what is the distance Matej can read the number of the bus from. We mark this distance as x. Because 1 arcminute is a very small angle (α), the resolution is very small compared to the distance from the bus and we can write

$$\alpha \approx \frac{y}{x}\,.$$

From this we obtain

$$x \approx \frac{y}{\alpha} \doteq 68.75\,\mathrm{m}\,.$$

Now let's calculate the time at which Matej reads the sign. For this we need to know the constant deceleration of the bus. The initial distance of the bus stop is $s = 50\,\mathrm{m}$. From the basic relations for constant acceleration we get

$$s = \frac{at_1^2}{2} = \frac{ut_1}{2}\,, \quad t_1 = \frac{2s}{u} = 10\,\mathrm{s}\,, \quad a = \frac{u^2}{2s} = 1\,\mathrm{m{\cdot}s^{-2}}\,.$$

As a part of the above calculation we also reached the time the bus needed to stop. The initial distance between Matej and the bus is $2s$. We are looking for a time it takes this distance to become x, i. e. decreases by $2s - x$. Using the same relations

$$vt_\mathrm{x} + ut_\mathrm{x} - \frac{1}{2}at_\mathrm{x}^2 = 2s - x\,, \quad \frac{1}{2}at_\mathrm{x}^2 - (u+v)\,t_\mathrm{x} + (2s - x) = 0\,,$$

$$t_\mathrm{x} = \frac{u + v - \sqrt{(u+v)^2 - 2a\,(2s-x)}}{a} \doteq 3.305\,\mathrm{s}\,.$$

In this time Matej walked $s_1 = vt_\mathrm{x} \doteq 3.67\,\mathrm{m}$.[3] He will run he remaining $s_2 = s - s_1 \doteq 46.33\,\mathrm{m}$ in $t_2 = \frac{s_2}{3v} \doteq 13.90\,\mathrm{s}$. If we add to that the time he walked, we have $t_2 + t_\mathrm{x} \doteq 17.20\,\mathrm{s}$. If we know from before that it took the bus $10\,\mathrm{s}$ to stop, and then another $t_0 = 10\,\mathrm{s}$ to wait on the bus stop, we can deduce Matej's reserve is comfortable $2.8\,\mathrm{s}$ so he manages to board the bus without any issues.

[3] When solving the quadratic equation, we chose the solution with minus sign, because we care about the shortest time, at which bus and Matej get into that distance. The solution with the plus sign would correspond to the situation when the bus slows all the way down to zero and starts accelerating backwards.

In this type of the question, we are forced to make many intermediate steps, and so, for the sake of time, it's useful to look for specific numerical values and not for general formulas, that could be quite lengthy. On the other hand we need to record each value with enough of a precision (3 or 4 significant digits), to stop the inaccuracies from accumulating and propagating to the result. General solution would look like this:

$$\begin{aligned} t &= t_0 + t_1 - t_2 - t_{\mathrm{x}} = t_0 + \frac{2s}{u} - \frac{s - vt_{\mathrm{x}}}{3v} - t_{\mathrm{x}} = \\ &= t_0 + \frac{2s}{u} - \frac{s}{3v} - \left(1 - \frac{v}{3v}\right)\frac{u + v - \sqrt{(u+v)^2 - 2a(2s - x)}}{a} = \\ &= t_0 + \frac{2s}{u} - \frac{s}{3v} - \frac{4s}{3}\frac{u + v - \sqrt{(u+v)^2 - \frac{u^2}{s}(2s - \frac{y}{\alpha})}}{u^2}\,, \end{aligned}$$

what definitely isn't a beautiful formula.

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Problem EC ... merry-go-flat

Matěj (with mass m) discovered a new equipment at a playground. It is a massive flat merry-go-round in the shape of a homogeneous disc with radius R and mass $2m$, which can rotate around its vertical axis without friction. Matěj stands on its edge and speeds up to velocity v (the velocity of the edge and Matěj with respect to the ground; Matěj is at rest with respect to the merry-go-round). Find the amount of work done by Matěj to move to the centre of the disc. Matěj can be approximated by a point mass. *Matěj likes to turn around.*

Matěj does work against the centrifugal force when he moves towards the centre. The entire merry-go-round is thus accelerated. Using the conservation of angular momentum, we can determine the work done by the difference in energies before and after Matěj moved. The initial kinetic energy is

$$E_0 = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2\,,$$

where $\omega = \frac{v}{R}$ is the initial angular velocity of the merry-go-round and $J = \frac{1}{2}\left(2m\right)R^2$ is the moment of inertia of a disc.The initial angular momentum is the sum of the angular momenta of the disc and of Matěj

$$L = J\omega + mRv\,.$$

When Matěj reaches the centre, his moment of inertia will be zero (his distance to the axis will be zero), but the total angular momentum is conserved, so the merry-go-round will accelerate to $\omega' > \omega$

$$L = J\omega'\,.$$

From the equality of angular momenta ω'

$$\omega' = \omega + \frac{mRv}{J}$$

and calculating the final kinetic energy

$$E_1 = \frac{1}{2}J\omega'^2 = \frac{1}{2}J\omega^2 + \omega mRv + \frac{1}{2}\frac{m^2R^2v^2}{J}\,.$$

The difference in energies before and after the move is equal to the work done

$$W = E_1 - E_0 = mv^2 + \frac{1}{2}\frac{m^2R^2v^2}{J} - \frac{1}{2}mv^2 = \frac{1}{2}mv^2 + \frac{m^2v^2}{2m}\,.$$

The work done is therefore $W = mv^2$.

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Problem ED ... finally a pleasant temperature

A planet orbits around a star with orbital velocity $v = 1.0\,\mathrm{km\cdot s^{-1}}$ on a circular trajectory. The mass of the star is $M_{\cdot} = 5.0\cdot 10^{30}\,\mathrm{kg}$ and its radius is $R_{\cdot} = 1.0\cdot 10^{7}\,\mathrm{km}$. The temperature of the planet is $T = 100\,\mathrm{K}$. Suppose that the planet and the star are both perfect black bodies. What is the temperature of the star in Kelvins? *Danka was cold.*

We can find distance r between the planet and the star from the equilibrium of the gravitational and the centrifugal force .

$$r = \frac{GM_{\cdot}}{v^2}\,.$$

Black body radiates energy

$$I = \sigma T^4\,.$$

so for the thermal equilibrium of the planet, energy received from the star is constantly being irradiated:

$$\sigma T_{\cdot}^4 4\pi R_{\cdot}^2 \frac{\pi R_p^2}{4\pi r^2} = \sigma T_p^4 4\pi R_p^2$$

For the temperature of star we get

$$T_H = T_p\sqrt{\frac{2r}{R_H}} = T_p\sqrt{\frac{2GM_H}{R_H v^2}} \doteq 25{,}800\,\mathrm{K}\,.$$

The temperature of the star is 25,800 K.

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Problem EE ... repulsive

There are 1,000 point masses in an otherwise empty plane. For each $k = 0, \dots, 999$, the coordinates of the kth point are $x_k = (k/1{,}000)^5\,\mathrm{km}$ and $y_k = (k/1{,}000)^3\,\mathrm{m}$. Each point has mass $M = 300\,\mathrm{kg}$. Find the value of the positive *electric charge of every particle (each point should have this charge) required for the system to be in equilibrium. The point masses are initially at rest.* *Lukáš was watching (self)restraining of naked singularities.*

The difficult looking task has a simple solution. Between each pair of particles k and l acts the sum of the gravitational and electrostatic force

$$F_{kl} = -\frac{GM^2}{r_{kl}^2} + \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{r_{kl}^2}\,.$$

Balance means that the sum of forces acting to each particles is zero, that is, $F_{kl} = 0$ for all k, l. Because both forces are decreasing with distance in the same way we meet the required condition by equality of the coefficients of the forces

$$GM^2 = \frac{1}{4\pi\varepsilon_0}Q^2 \,,$$
$$Q = M\sqrt{4\pi\varepsilon_0 G} \doteq 2.6 \cdot 10^{-8}\,\mathrm{C}\,.$$

Each of the points masses must be charged by the electric charge $Q \doteq 2.6 \cdot 10^{-8}$ C.

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Problem EF ... hang up that pendulum!

Consider a thin, rigid, homogeneous rod. At a certain distance along the length of the rod, we drill in a small hole in such a way that the period of small oscillations of the rod around a perpendicular axis passing through that hole is the minimum possible. The hole separates the rod into two segments. What is the ratio of lengths of these segments?

Matěj would like to hang Jáchym.

We use the formula for the period of a physical pendulum

$$T = 2\pi\sqrt{\frac{J + ml^2}{mgl}}\,,$$

where m is the mass of the pendulum, l is the distance between the center of gravity and the point of suspension and J is moment of inertia to the axis passing through the center of gravity. For homogeneous rod with length L, we have $J = \frac{1}{12}mL^2$. The moment of inertia and the mass is constant. We are looking for l such that the expression inside the square root is minimal.

$$\frac{J + ml^2}{ml} = \frac{\frac{1}{12}L^2 + l^2}{l}$$

We use a first derivative

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{L^2}{12l} + l\right) = -\frac{L^2}{12l^2} + 1\,.$$

By letting this equal to zero we get

$$l = \frac{L}{\sqrt{12}}\,.$$

The required ratio is

$$\frac{\frac{1}{2} + \frac{1}{\sqrt{12}}}{\frac{1}{2} - \frac{1}{\sqrt{12}}} = \frac{\sqrt{3} + 3}{3 - \sqrt{3}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3} \doteq 3.732\,.$$

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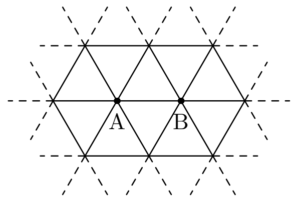
Problem EG ... triangular network

Consider an infinite network of resistive wire depicted in the figure. What will be the resistance between adjacent points A and B if the wire with length |AB| has resistance R_1?

Karel would like to have an infinite network at home.



The problem is best solved by a trick, just as most of those which have an infinite amount of resistors and some type of symmetry.

We use the superposition principle, where a combination of solutions of partial problems is a solution to the whole problem. So although we can't find the current between the two points directly, we can find the current from the first point to infinity and from infinity to the other point. First let us consider there is an electric current I from outside the grid entering at A and exiting at B.

So in the first case, we have current I going to point A, from the source. Because of the 6-fold symmetry, we know that the current through each of the six wires meeting at A must be $I/6$. The same result is when we calcuate the current from infinity to B, where the directions of the currents are pointing inwards this time. The solution for the current from point A to B es equal to the one sending the curren from A to infinity and then from infinity to B. So we see that the total current going through A–B is the sum of the two currents, which is equal to $I/3$.

We don't really care about other wires than the connecting one, so we have no idea about the value of the electric current there, but it's not important for our solution.

To find the resistance between, we use $U_{\mathrm{AB}} = R_1 I/3 = RI$, which solved for R gives $R = R_1/3$.

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Problem EH ... zero to hundred for the third time

One of the performance characteristics of a car is the time needed to accelerate from $0\,\mathrm{km\cdot h^{-1}}$ to $100\,\mathrm{km\cdot h^{-1}}$. Consider a car that can do it in $4.0\,\mathrm{s}$. Find the distance travelled by the car while accelerating under the assumption of constant jerk of the car. Jerk (often denoted by j) is the rate of change (time derivative) of acceleration, just like acceleration is the rate of change of velocity.

Karel was wondering about acccelerating.

We start with the basic relationship between time-dependent acceleration and time elapsed from the start

$$a = jt\,.$$

We can derive the formula for time-dependent velocity by integrating previous formula

$$v = \int_0^t jt' \mathrm{d}t' = \frac{1}{2} jt^2\,.$$

Now, we are able to calculate the jerk j

$$j = \frac{2v}{t^2} = 3.47\,\mathrm{m\cdot s^{-3}}\,.$$

We need to integrate once again in order to calculate the travelled distance.

$$s = \int_0^t \frac{1}{2} j t'^2 \mathrm{d}t' = \frac{1}{6} j t^3 = \frac{1}{3} v t = 37.0\,\mathrm{m}\,.$$

By the way, this distance is $\frac{1}{3}$ shorter than the distance in the case of constant acceleration.

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Problem FA ... stretched

Consider a rubber band (not a loop, just a single strip) with density ϱ made of a material with Young modulus E. Its rest length (for example when lying freely on a table) is L. We suspend the rubber band from one of its ends. What length will it have now?

Matěj was playing with a rubber band.

Each small part of the rubber band of a length of $\mathrm{d}l$ stretches to a new length $\mathrm{d}x$ given by

$$\mathrm{d}x = \mathrm{d}l \left(1 + \frac{\sigma}{E}\right) ,$$

where σ is the tension at that point. Let's suppose the cross section of the rubber band is S. The tension is then given by

$$\sigma = \frac{F}{S} ,$$

where F is the force pulling the rubber band downwards. $F = Sl\varrho g$, where l is the distance of the point from the end of the rubber band in the **unstretched** state (otherwise the density wouldn't be constant). Therefore

$$\mathrm{d}x = \left(1 + \frac{l\varrho g}{E}\right) \mathrm{d}l\,.$$

$$L_1 = \int_0^{L_1} \mathrm{d}x = \int_0^{L} \left(1 + \frac{l\varrho g}{E}\right) \mathrm{d}l = L + \frac{L^2 \varrho g}{2E}\,.$$

The solution can be also found noticing that the extension depends linearly on the length of the length under the point, which implies the extension will be the same as if we put a point mass on the end of the rubber band of half of of weight of the rubber band and ignore the weight of the band itself.

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Problem FB ... procrastination

Matěj likes to spend time watching videos on YouTube instead of working. YouTube uses a sophisticated algorithm to suggest videos that you might be interested in after you finish watching the current one. This algorithm is so good that with probability $p = 80\%$, Matěj likes *one of the suggested videos and starts watching it. The average length of one video is* $t = 7\,\mathrm{min}$*. How long will it take on average for Matěj to stop watching videos and start working on this problem?*

The following formulas can be handy: $\sum_{i=1}^{\infty} ix^i = \frac{x}{(x-1)^2}$ *for* $|x| < 1$, $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Mezera: I'll write something here later.

Suggestion are independent, so the probability of watching nth video is

$$p_\mathrm{n} = p^{n-1}\,.$$

Probability, that he stops after nth video is

$$P_\mathrm{n} = p' p_\mathrm{n} = (1-p)\,p^{n-1}\,.$$

and it takes nt of time. For average time, we weight each of the possible watching times with their probability:

$$t_\mathrm{p} = \sum_{\mathrm{n}=1}^{\infty} (1-p)\,p^{n-1} nt = \frac{1-p}{p} t \sum_{\mathrm{n}=1}^{\infty} np^n = \\ = \frac{1-p}{p} t \frac{p}{(1-p)^2} = \frac{t}{1-p} = 35\,\mathrm{min}\,.$$

Matěj stops procrastinating after $35\,\mathrm{min}$ on average.

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Problem FC ... a lot of computations

Matěj bought a digital clock which displays time in the 24-hour format. Each digit is formed by up to seven lit diodes (28 diodes in total). Each diode consumes $0.1\,\mathrm{mW}$ *of power. Matěj placed four new AAA batteries in the clock, set the time to 12:00 immediately and hung the clock on the wall. Some time later, he looked at the clock and saw the time for a moment, but suddenly, the clock discharged and switched off. Determine the time Matěj saw. The capacity of one AAA battery is* $2.5\,\mathrm{Wh}$*; energy is only used to power the diodes, there are no other losses.*
Note: *Digits six and nine are formed by six diodes each, there's no separator on the clock and all four digits are displayed all the time. All the values are perfectly exact.*

Matěj wanted to make a simple problem which requires computing a lot of stuff.

Clocks can work for months or even years with a single set of batteries. We shall now calculate how much energy this clock uses per day and will then be interested only in the remainder after dividing by this daily power consumption. We can assume that that the change of displayed digits (switching diodes on and off) is instantaneous. We will split the display to specific digits and determine their average daily power usage.

The last digits cycles digits 0 to 9, giving mean power of this digit

$$P_1 = \frac{6+2+5+5+4+5+6+3+7+6}{10} 0.1\,\mathrm{mW} = 0.49\,\mathrm{mW}\,.$$

The ten minutes digit cycles between 0 and 5, so its mean power is

$$P_2 = \frac{6+2+5+5+4+5}{6} 0.1\,\mathrm{mW} = 0.45\,\mathrm{mW}\,.$$

The second digit cycles goes through 0 to 9 twice a day and then goes through 0, 1, 2, 3. This yields a mean of

$$P_3 = \frac{2\,(6+2+5+5+4+5+6+3+7+6)+6+2+5+5}{24} 0.1\,\mathrm{W} = \frac{29}{60}\,\mathrm{mW}\,.$$

The first digits shows 0 for for the first 10 hours, then 1 for 10 hours and finally 2 for the remaining 4 hours

$$P_4 = \frac{10\cdot 6 + 10\cdot 2 + 4\cdot 5}{24} 0.1\,\mathrm{mW} = \frac{25}{60}\,\mathrm{mW}\,.$$

The clock's daily power use is therefore

$$E_1 = (P_1 + P_2 + P_3 + P_4)\cdot 24\,\mathrm{h} = 44.16\,\mathrm{mWh}\,.$$

The remainder after dividing the battery capacity by this value is

$$E_\mathrm{z} = \left(2\cdot 5\cdot 10^3 \mod 44{,}16\right)\,\mathrm{mWh} = 19.84\,\mathrm{mWh}\,.$$

It is now 12:00 on the clock and the remaining battery capacity is E_z. That's less than half of the daily power consumption, so we try to subtract the power used in the next 10 hours:

$$\begin{aligned} E_\mathrm{d} &= (P_1 + P_2)\cdot 10\,\mathrm{h} + P_1\cdot 10\,\mathrm{h} + 8\cdot 2\cdot 0.1\,\mathrm{mWh} + 2\cdot 5\cdot 0.1\,\mathrm{mWh} \\ &= 16.90\,\mathrm{mWh}\,. \end{aligned}$$

The clock is now at 22:00 and we only have $19.84\,\mathrm{mWh} - 16.90\,\mathrm{mWh} = 2.94\,\mathrm{mWh}$ in the batteries left. During the twenty-second hour, the clock uses

$$E_{22} = (P_1 + P_2)\cdot 1\,\mathrm{h} + 5\cdot 0.1\,\mathrm{mWh} + 5\cdot 0.1\,\mathrm{mWh} = 1.94\,\mathrm{mWh}\,.$$

Remaining energy is now $2.94\,\mathrm{mWh} - 1.94\,\mathrm{mWh} = 1.00\,\mathrm{mWh} = 60.0\,\mathrm{mWmin}$. We have now converted the units to the unusual milliwatt-minutes and we will numerically count the minutes. It is 23:00, in the next "30 min" the clock uses

$$\begin{aligned} E_{30\mathrm{min}} &= P_1\cdot 30\,\mathrm{min} + (6+2+5)\cdot 0.1\,\mathrm{mW}\cdot 10\,\mathrm{min} \\ &\quad + 5\cdot 0.1\,\mathrm{mW}\cdot 30\,\mathrm{min} + 5\cdot 0.1\,\mathrm{mW}\cdot 30\,\mathrm{min} \\ &= 57.7\,\mathrm{mWmin}\,. \end{aligned}$$

It is now 23:30 with $60.0\,\mathrm{mWmin} - 57.7\,\mathrm{mWmin} = 2.3\,\mathrm{mWmin}$ left. We find out that during the next minute, the battery loses

$$\begin{aligned} E_{1\mathrm{min}} &= 6\cdot 0.1\,\mathrm{mWmin} + 5\cdot 0.1\,\mathrm{mWmin} + 5\cdot 0.1\,\mathrm{mWmin} + 5\cdot 0.1\,\mathrm{mWmin} \\ &= 2.1\,\mathrm{mWmin}\,. \end{aligned}$$

leaving us with $0.2\,\mathrm{mWmin}$ and we can easily verify that this will not last for another minute. The clock switches off at 23:31.

This problem does not have a general solution, the only way to solve it is this numerical calculation, but we did not round any intermediate values, so our result is exact. In reality we only know the battery capacity and diode power with a limited precision, which would surely be insufficient to determine the exact time when the clock looses power as we frequently subtracted remaining capacity, which would accumulate the initial uncertainties. We would get a very imprecise result.

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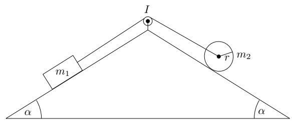
Problem FD ... roof and cylinder



There are two objects placed on a roof, a cuboid and a homogeneous cylinder, connected by a rope. The roof is formed by two inclined planes with inclination angle $\alpha = 30.0^\circ$ with respect to the horizontal plane. The rope passes through a pulley with the moment of inertia $I = 0.100\,\mathrm{kg{\cdot}m^2}$ and radius $r_\mathrm{k} = 0.100\,\mathrm{m}$. The mass of the cuboid is $m_1 = 5.00\,\mathrm{kg}$, the radius of the cylinder's base is $r_\mathrm{v} = 0.30\,\mathrm{m}$ and its mass is $m_2 = 10.0\,\mathrm{kg}$. What will be the acceleration of the cuboid (including its direction)? The coefficient of friction between either object and the roof is $f = 0.50$. Neglect the rolling resistance. *Karel modified another problem.*

First, let's analyse the forces and determine in which direction (if at all), will the system move.

For the bodies to move, the total force acting on the cuboid must exceed the force of frictionThe cuboid is pulled to one side by the parallel component of the force of gravity and pulled the other side by the force due to the cylinder

$$F_2 = m_1 g \sin\alpha - m_2 g \sin\alpha = (m_1 - m_2)\, g \sin\alpha = -24.53\,\mathrm{N}\,.$$

Looking at the numbers we see the cuboid will move upward, as the force of friction is exceeded. The entire system is therefore accelerated by the force

$$F_0 = F_2 - F_\mathrm{t} = -(m_1 - m_2)\, g\sin\alpha - f m_1 g \cos\alpha = 3.29\,\mathrm{N}\,.$$

This force must accelerate the cuboid and the cylinder and at the same time, spin up the cylinder and the pulley. If a force F acts on a body with moment of inertia J at distance r from the axis (perpendicular to the axis and r), from the change of angular momentum we find the expression for angular acceleration $\varepsilon = \frac{a}{r}$

$$F_0 r = J\varepsilon\,,\quad F_0 = \frac{J}{r^2} a\,.$$

Denoting acceleration a and using balance of the forces

$$F_0 = (m_1 + m_2)\, a + \frac{I}{r_\mathrm{k}^2} a + \frac{I_\mathrm{v}}{r_\mathrm{v}^2} a\,,$$

where $I_\mathrm{v} = m_2 r_\mathrm{v}^2/2$ is the moment of inertia of the cylinder.

$$F_0 = \left(m_1 + \frac{3}{2} m_2 + \frac{I}{r_\mathrm{k}^2}\right) a\,,$$

$$a = \frac{(m_2 - m_1)\sin\alpha - f m_1 \cos\alpha}{m_1 + \frac{3}{2} m_2 + \frac{I}{r_\mathrm{k}^2}} g = 0.110\,\mathrm{m{\cdot}s^{-2}}\,.$$

The cuboid will move with acceleration $0.110\,\mathrm{m\cdot s^{-2}}$ upward.

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Problem FE ... zero to hundred for the second time

One of the performance characteristics of a car is the time needed to accelerate from $0\,\mathrm{km\cdot h^{-1}}$ to $100\,\mathrm{km\cdot h^{-1}}$. Consider a car that can do it in $4.0\,\mathrm{s}$. Find the distance travelled by the car while accelerating. Assume that the power of the car's motor is constant.

Karel was wondering about acccelerating.

The power is equal to the change of kinetic energy over time

$$P = \frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}t} = \frac{\mathrm{d}\frac{1}{2}mv^2}{\mathrm{d}t} = mva = m\dot{x}\ddot{x}\,,$$

where the mass of the car is m, the instantaneous velocity is v and the instantaneous acceleration is $a = \ddot{x}$. In this case, it is easiest to integrate

$$P = \frac{\mathrm{d}\frac{1}{2}m\dot{x}^2}{\mathrm{d}t} \quad \Rightarrow \quad Pt + E_0 = \frac{1}{2}m\dot{x}^2\,,$$

where the integration constant is E_0; it corresponds to initial kinetic energy, so it is equal to 0. Now, we express the velocity before integrating again

$$\dot{x} = \sqrt{\frac{2P}{m}}t^{\frac{1}{2}} \quad \Rightarrow \quad x = \frac{2}{3}\sqrt{\frac{2P}{m}}t^{\frac{3}{2}}\,.$$

We get

$$\sqrt{\frac{2P}{m}} = \frac{\dot{x}}{t^{\frac{1}{2}}}\,.$$

Combining it with the expression for the distance, we get $x = \frac{2}{3}vt \doteq 74.1\,\mathrm{m}$.

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Problem FF ... a ring

Consider a non-homogeneous rod with length L and weight M suspended from one of its ends. The distance of the centre of mass of the rod from the point of suspension is l and the corresponding moment of inertia with respect to that point of suspension is J. We place a small ring of negligible dimensions on the rod so that the frequency of oscillations of the rod with the ring is twice the frequency of oscillations of the original rod. Let's denote the weight of the ring by m and its distance from the point of suspension by x. Find the weight m such that the distance x is unambiguous and express this value of x.

Kuba was wondering which solution to choose.

Halved oscillation velocity corresponds to a double period or a half angular frequency. We calculate the angular frequency from the formula

$$\omega = \sqrt{\frac{Mgl}{J}}\,.$$

The total moment of inertia is additive, so after the placement of the ring we have

$$J^* = J + mx^2 \,.$$

Furthermore, we have an equation for the new position of the centre of gravity

$$Ml + mx = (M + m)\, l^* \,.$$

Now, we can write for the new angular frequency

$$\omega^* = \sqrt{\frac{(M+m)\, gl^*}{J^*}} = \sqrt{\frac{g\,(Ml + mx)}{J + mx^2}} \,.$$

By solving the equation $\omega^* = 2\omega$ we get

$$x = \frac{J}{8Ml}\left(1 \pm \sqrt{1 - \frac{48l^2M^2}{Jm}}\right) \,.$$

We see that both solutions degenerate into one if the square root is zero for which we have

$$m = \frac{48M^2l^2}{J} \,.$$

In this case, we get an equation for the position of the ring

$$x = \frac{J}{8Ml} \,,$$

which is the desired result.

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Problem FG ... Matěj's four spheres

Four identical homogeneous spheres are piled up on a horizontal surface made from the same material as are the spheres. The centres of the spheres form a regular tetrahedron (three spheres are lying on the surface and form an equilateral triangle, the fourth is placed on top of them). Find the minimum possible value of the coefficient of static friction between the surfaces necessary for the spheres to remain at rest. *Matěj was arranging oranges.*

Thanks to the symmetry of the problem, it's sufficient to examine forces acting on only one bottom ball. Top ball exerts force F onto the bottom ball. Let's decompose F in directions parallel (F_{t}) and perpendicular (F_{n}) to surface of the balls at the point of contact. From the geometry of a tetrahedron, for an angle φ between F_{n} and horizontal plane, the following applies:

$$\tan\varphi = \sqrt{2} \,.$$

For the ball to remain in rest in is necessary that:

- Total torque is zero. Therefore friction between the bottom ball and the base has the same magnitude as F_{t} (in an opposite direction).

- Total force is zero. Equilibrium has to be achieved in both horizontal and vertical direction. We get

$$F_\mathrm{n}\cos\varphi - F_\mathrm{t}\sin\varphi - F_\mathrm{t} = 0\,,$$
$$F_\mathrm{n}\frac{\cos\varphi}{\sin\varphi + 1} = F_\mathrm{t}\,.$$

Balls stay intact, if

$$fF_\mathrm{n} \geq F_\mathrm{t}\,,$$
$$f \geq \frac{\cos\varphi}{\sin\varphi + 1} = \frac{1}{\sqrt{2}+\sqrt{3}} = \sqrt{3} - \sqrt{2} \doteq 0.318\,.$$

In that scenario, the downward force exerted on the base through the single ball is

$$4F_g/3 \geq fF_\mathrm{n} \geq F_\mathrm{t}$$

so the friction between the bottom ball and the base is high enough.

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Problem FH ... spinning container

Consider a cylindrical container with base radius $R = 10\,\mathrm{cm}$ containing $V = 2\,\mathrm{l}$ of water. Calculate the minimum height of the container such that we avoid spilling any water when we spin it around its axis for a long time with angular velocity $\omega = 5\,\mathrm{rad{\cdot}s^{-1}}$.

Matěj likes to spin stuff.

After a certain time, the water converges to a state, where it rotates with the same angular velocity as the container. Our task is to find a function describing the height of the surface depending on the distance from the rotational axis. The basic idea is that the water surface is equal to some equipotential, because the water adapts the shape with minimum potential energy. Knowing that in the vertical direction there is a constant (gravitational) acceleration and in the horizontal direction the acceleration is proportional to the distance from the centre, we find the water surface to be a rotational paraboloid.

Let's call the function giving the height of the surface from the radial distance $h(r)$. Because the surface must be perpendicular to the force (therefore also the acceleration from Newtons second law), the slope of the function is equal to the ratio of the horizontal and vertical acceleration.

$$\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{\omega^2 r}{g}\,.$$

Through integration we get $h(r)$

$$h(r) = \int \mathrm{d}h = \int \frac{\omega^2 r}{g}\mathrm{d}r = \frac{\omega^2 r^2}{2g} + C\,,$$

C is an integration constant, the value of which we get from the initial conditions – in this case from the volume V, which remains constant. We split the water into thin cylindrical rings with a volume of $2\pi r h(r)\mathrm{d}r$.

$$V = \int_0^R 2\pi r h(r)\mathrm{d}r = 2\pi \int_0^R \left(\frac{\omega^2 r^3}{2g} + Cr\right)\mathrm{d}r = \frac{\pi\omega^2 R^4}{4g} + C\pi R^2\,,$$

$$C = \frac{V}{\pi R^2} - \frac{\omega^2 R^2}{4g}\,.$$

The value of $h(r)$ in R gives us the lowest height, which is able to keep the water from spilling.

$$h(R) = \frac{\omega^2 R^2}{4g} + \frac{V}{\pi R^2} = 0.070\,\mathrm{m}\,.$$

It should be noted, that our model does not explicitly include the bottom of the container, so we have to be sure it does not intersect with the paraboloid i.e $h(r)$ is non-negative, which would give us incorrect results. This can be shown to hold in this case.

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Problem GA ... rope

There's a coil of rope with linear density ϱ lying on the ground. We pick up one end of the rope and lift it upwards to height h with constant velocity v. What's the difference between the work we need to perform to do this and the sum of potential and kinetic energy of the ordered motion of the rope at the end of the process?

Hint: *It really isn't zero!* *Jáchym felt like he lacked some energy.*

As long as some mass of the rope m is in the air it is affected by the gravitational force $F_G = -mg$. We have to act by force F which we determine from the second Newtonian law as

$$F + F_G = \dot{p} = m\dot{v} + \dot{m}v = \dot{m}v\,. \tag{2}$$

Here we used the fact that $\dot{v} = 0$ because the lifting speed is constant. For some time $\mathrm{d}t$ we lift $\mathrm{d}x = v\mathrm{d}t$ of rope which gives us an equation for the time derivative of velocity

$$\dot{m} = \varrho v\,.$$

Through substituting this equation into the equation (2) we get

$$F = \dot{m}v - F_G = \varrho v^2 + mg = \varrho v^2 + x\varrho g\,,$$

where x means the length of rope that has been already raised. To compute the work, we have to integrate this equation from zero to h

$$W = \int_0^h F\mathrm{d}x = \int_0^h \varrho v^2 + x\varrho g\mathrm{d}x = \left[x\varrho v^2 + \frac{1}{2}x^2\varrho g\right]_0^h = h\varrho v^2 + \frac{1}{2}h^2\varrho g\,.$$

By lifting the end of the rope up to the height h we give it kinetic and potential energy

$$E = \frac{1}{2} h \varrho v^2 + \frac{1}{2} h^2 \varrho g\,,$$

which leads to the energy "loss"

$$\Delta E = W - E = \frac{1}{2} h \varrho v^2\,.$$

The energy itself is, of course, not lost, only transformed to the energy of the oscillations of the rope. In this way of lifting small parts of the rope of length dx are accelerated to the speed v in the zero trajectory and for zero time which of course is not possible. Real rope is flexible so the small parts accelerate gradually. If we assumed that the rope is perfectly rigid so that the oscillations and losses of energy can not happen we would realize that the rope could not be lifted in this way.

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Problem GB ... solar rod

Consider an infinitely long, thin semiconducting strip with width $1\,\mathrm{cm}$. *The area conductivity of the semiconducting material is directly proportional to illuminance with proportionality constant* $\alpha = 0.03\,\mathrm{S{\cdot}lx^{-1}}$. *We place a point source of light with luminous intensity* $2\,\mathrm{cd}$ *at height* $1\,\mathrm{m}$ *above the axis of the semiconductor. Then, we connect two perfectly conductive infinitely long electrodes to the two infinite edges of the strip. The voltage difference between the electrodes is* $7\,\mathrm{V}$. *What current (in Amperes) will flow between the electrodes?*

Mikuláš likes it when others have to solve unpleasant integrals.

The illumination of the semiconductor decreases both quadratically with distance and with the cosine of the angle of incidence of the rays to the semiconductor so generally according to the formula

$$\frac{hI}{\left(\sqrt{h^2 + x^2}\right)^3}\,,$$

where h is the height of the lamp above the semiconductor, I is luminous intensity of the source and x is the distance from the centre of the semiconductor. The specific resistance is then given by the formula

$$\frac{\left(\sqrt{h^2 + x^2}\right)^3}{hI\alpha}$$

and the resistance of the element of length dx is

$$\frac{y\left(\sqrt{h^2 + x^2}\right)^3}{hI\alpha \mathrm{d}x}\,,$$

where y is the width of the strip. The inverse quantity of resistance is obtained by integrating the inverse quantity of resistivity because it behaves like a parallel connection of an infinite number of resistors.

$$\frac{1}{R} = \int_{-\infty}^{\infty} \frac{hI\alpha}{y\left(\sqrt{h^2 + x^2}\right)^3} \mathrm{d}x,$$

We solve by hyperbolic substitution

$$\frac{1}{R} = \frac{I\alpha}{hy}\left[\frac{x}{\sqrt{x^2+h^2}}\right]_{-\infty}^{\infty} = \frac{2I\alpha}{hy}\,.$$

The current is then obtained according to the formula

$$I_{\mathrm{El}} = \frac{U}{R} = \frac{2UI\alpha}{hy}\,.$$

By substitution in the equation we get $I_{\mathrm{El}} = 84\,\mathrm{A}$.

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Problem GC ... fireworks

The organisers forgot to buy fireworks for the New Year's Eve, so they decided to build a copy of the Tsar bomba. Let's assume that $2\,\%$ of the energy released in the reaction ${}^2_1\mathrm{D} + {}^3_1\mathrm{T} \longrightarrow {}^4_2\mathrm{He} + \mathrm{n}^0 +$ is emitted as electromagnetic radiation in the form of one photon per reaction. How fast should the organisers run away in order to be able to watch the "fireworks" as visible light ($550\,\mathrm{nm}$)? Compute the difference $c - v$ between the speed of light and this velocity. The rest masses are $m({}^2_1\mathrm{D}) = 1{,}876.1\,\mathrm{MeV{\cdot}c^{-2}}$, $m({}^3_1\mathrm{T}) = 2{,}809.4\,\mathrm{MeV{\cdot}c^{-2}}$, $m({}^4_2\mathrm{He}) = 3{,}728.4\,\mathrm{MeV{\cdot}c^{-2}}$, $m(\mathrm{n}^0) = 939.6\,\mathrm{MeV{\cdot}c^{-2}}$. *We really forgot.*

The number of photons does not concern us, we can concentrate on just one reaction and the photon generated by it. The total energy ΔE released during this reaction can be calculated from the law of conservation of energy (we can assume that the particles entering the reaction have negligible kinetic energy) in the form

$$m({}^2_1\mathrm{D})c^2 + m({}^3_1\mathrm{T})c^2 = \Delta E + m({}^4_2\mathrm{He})c^2 + m(\mathrm{n}^0)c^2\,.$$

Here, $\Delta E \doteq 17.5\,\mathrm{MeV}$ includes the kinetic energy of all particles produced during the reaction. The energy of the photon (its total energy is the same as its kinetic energy because the rest mass of a photon is 0) according to the problem statement is

$$0.02\Delta E \doteq 0.35\,\mathrm{MeV}\,,$$

the wavelength of the emitted photon (in the centre of mass reference frame of the reaction) is therefore

$$\lambda = \frac{hc}{E} \doteq 3.5\,\mathrm{pm}\,.$$

When escaping from the "fireworks" with velocity v, the Doppler effect occurs and the observed frequency is given by the formula

$$\lambda' = \lambda\sqrt{\frac{c+v}{c-v}}\,.$$

Because the required wavelength λ' is much larger than λ, the velocity must be $v \approx c$ and we can approximate

$$\frac{\lambda'}{\lambda} \approx \sqrt{\frac{2c}{c-v}} \quad \Rightarrow \quad c - v \approx 2c\left(\frac{\lambda}{\lambda'}\right)^2\,.$$

We get $c - v \doteq 8 \cdot 10^{-11} c \doteq 0.024\,\mathrm{m{\cdot}s^{-1}}$ – we need to run away from the Tsar bomba at near-lightspeed.

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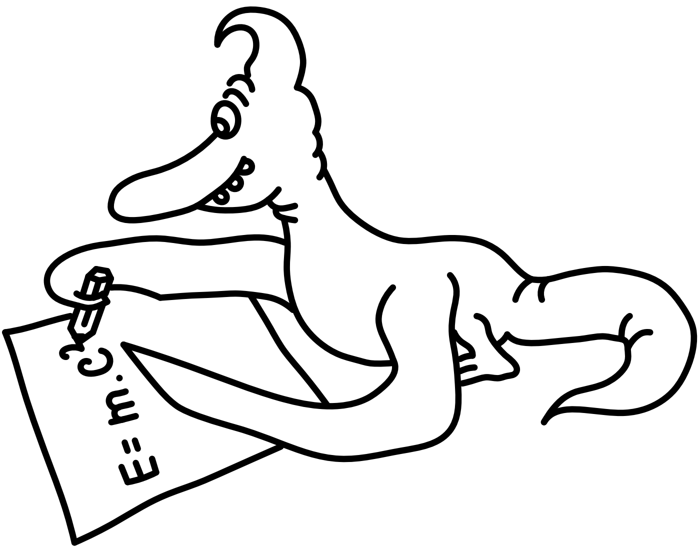

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